AD-A015 290

FLEXIBILITY AND DECISION ANALYSIS

Miley Wesson Merkhofer

Stanford University

Prepared for:

Office of Naval Research
National Science Foundation
Advanced Research Projects Agency

30 June 1975

DISTRIBUTED BY:

NTIS
National Technical Information Service
U. S. DEPARTMENT OF COMMERCE

Research Report No. EES-DA-75-1
June 30, 1975

FLEXIBILITY AND DECISION ANALYSIS

MILEY WESSON MERKHOFER

DECISION ANALYSIS PROGRAM

Professor Ronald A. Howard
Principal Investigator

DEPARTMENT OF ENGINEERING-ECONOMIC SYSTEMS
Stanford University
Stanford, California 94305

SPONSORSHIPS

Advanced Research Projects Agency, Human Resources Research Office, ARPA Order No. 2449, monitored by Engineering Psychology Programs, Office of Naval Research, Contract No. N00014-67-A-0112-0077 (NR197-024).

National Science Foundation, NSF Grant GK-36491.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: EES-DA-75-1	2. GOVT ACCESSION NO	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): "Flexibility and Decision Analysis"		5. TYPE OF REPORT & PERIOD COVERED: Technical 5/1/73 to 5/31/75
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Miley Wesson Merkhofer		8. CONTRACT OR GRANT NUMBER(s): N00014-67-A-0112-0077
9. PERFORMING ORGANIZATION NAME AND ADDRESS: The Board of Trustees of the Leland Stanford Junior University, c/o Office of Research Administrator, Encina Hall, Stanford, California 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 000000 ARPA Order #2449
11. CONTROLLING OFFICE NAME AND ADDRESS: Advanced Research Projects Agency Human Resources Research Office 1400 Wilson Blvd., Arlington, Virginia 22209		12. REPORT DATE: June 30, 1975
		13. NUMBER OF PAGES: 132 pages
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): Engineering Psychology Programs, Code 455 Office of Naval Research 800 No. Quincy Street, Arlington, Virginia 22217		15. SECURITY CLASS. (of this report): Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

A dissertation submitted to the Department of Engineering-Economic Systems and the Committee on Graduate Studies of Stanford University in partial fulfillment of the Requirements for the Degree of Doctor of Philosophy, June 1975.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

DECISION ANALYSIS	VALUE OF INFORMATION	QUANTIZATION
DECISION FLEXIBILITY	VALUE OF FLEXIBILITY	QUADRATIC PROBLEM

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The notion that a good decision strategy is a flexible one has long been intuitively appreciated by decision makers. Decision analysis, however, has had little to say on the subject of flexibility. The purpose of this thesis is to place the flexibility concept within the decision analysis framework.

The analysis begins with an application of decision theory techniques to the problem of choosing between flexible and inflexible decision strategies.

(Continued on reverse side)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Block 20 (continued)

A precise mathematical definition of decision flexibility is proposed wherein the relative flexibility of a decision is measured by the size of the decision choice set. Further application of the theory of decision analysis provides a measure of the value of flexibility.

A strong complementarity between information and flexibility is observed. The more information is expected during the execution of a plan, the more valuable is flexibility. Conversely, the more flexibility one has, the greater the value of information gathering. The concepts of the value of information and the value of flexibility are seen to be inseparable. We define a more fundamental concept of the value of information given flexibility and point out the applications of the concept to decision making.

The value of information given flexibility is analyzed for decision problems with quadratic value functions. The restriction to this class allows us to isolate the impact of two characteristics that in a large part determine tne economic impact of information and flexibility. These are (1) the extent of correlation among the state and information variables and (2) the amount of interaction among the decision variables.

A portion of the thesis is devoted to a study of the effects of the information and flexibility quantization necessary for a decision tree representation of a sequential decision problem. The analysis indicates that even a very rough system of quantization tends to introduce only small losses into the optimal solution of a sequential decision problem. Thus, support is given to the method of solution of sequential problems by decision trees.

Finally, methods are presented for simplifying the calculation of the value of information given flexibility through the use of sensitivity analysis.

ABSTRACT

The notion that a good decision strategy is a flexible one has long been intuitively appreciated by decision makers. Decision analysis, however, has had little to say on the subject of flexibility. The purpose of this thesis is to place the flexibility concept within the decision analysis framework.

The analysis begins with an application of decision theory techniques to the problem of choosing between flexible and inflexible decision strategies. A precise mathematical definition of decision flexibility is proposed wherein the relative flexibility of a decision is measured by the size of the decision choice set. Further application of the theory of decision analysis provides a measure of the value of flexibility.

A strong complementarity between information and flexibility is observed. The more information is expected during the execution of a plan, the more valuable is flexibility. Conversely, the more flexibility one has, the greater the value of information gathering. The concepts of the value of information and the value of flexibility are seen to be inseparable. We define a more fundamental concept of the value of information given flexibility and point out the applications of the concept to decision making.

The value of information given flexibility is analyzed for decision problems with quadratic value functions. The restriction to this class allows us to isolate the impact of two characteristics that in a large part determine the economic impact of information and flexibility. These are (1) the extent of correlation among the state and information variables and (2) the amount of interaction among the decision variables.

A portion of the thesis is devoted to a study of the effects of the information and flexibility quantization necessary for a decision tree representation of a sequential decision problem. The analysis indicates that even a very rough system of quantization tends

to introduce only small losses into the optimal solution of a sequential decision problem. Thus, support is given to the method of solution of sequential problems by decision trees.

Finally, methods are presented for simplifying the calculation of the value of information given flexibility through the use of sensitivity analysis.

ACKNOWLEDGEMENTS

I wish to acknowledge and thank my adviser, Professor Ronald A. Howard, for his advice and guidance throughout the course of this research. I also wish to thank Professor Edward J. Sondik and Dr. James E. Matheson for their helpful comments and constructive criticism as readers, Mrs. Louise Goodrich for her editorial help, and Mrs. Ditter Peschcke-Koedt for typing the manuscript.

For their encouragement, as well as intellectual and moral support, I owe a special debt of gratitude to fellow students Thomas R. Rice and Hans W. Wynholds.

Finally, I am indebted to the National Science Foundation for financial support during my early years of graduate study. This research was partially supported by the Advanced Research Projects Agency of the Department of Defense, as monitored by the Office of Naval Research under Contract No. N00014-67-A-0112-0077. This research was also partially supported by the National Science Foundation under NSF Grant GK-36491.

TABLE OF CONTENTS

CHAPTER | page

LIST OF FIGURES

Figure		page

LIST OF TABLES

INTRODUCTION AND SUMMARY OF RESULTS

In the course of the scientific development of the theory of decision-making, it often has been the case that substantial improvements in understanding have resulted from the mathematical refinement of common terms. Words such as "uncertainty," "risk," and "information" have everyday meanings that fail to distinguish adequately a problem's characteristics. Consequently for the purposes of economic analysis, their definitions have been narrowed, or they have been replaced by new terms whose meanings are more precise.

The objective of this thesis is to present a mathematically precise definition of the term "flexibility" appropriate to the discipline of decision analysis. We believe that an important measure of the appropriateness of a definition should be the worth of the insights generated from its use. Consequently, a major portion of our work is an exploration for useful results that may be derived from the application of our definition.

Chapter 1

Chapter 1 presents our definition of decision flexibility. We view the flexibility of a given decision variable to be determined by the size of the choice set associated with that variable. Roughly speaking, the larger the choice set--that is, the more alternatives that are available for a decision--the greater is the decision flexibility.

Since flexibility is a property of the choice set for a variable rather than a property of the variable itself, the degree of flexibility possessed by each decision will change during the decision process. The actions taken early in a decision process often affect the number of alternatives available later on. The action to acquire a choice set is a decision that produces flexibility on another variable. Thus, the decision to keep a large amount of cash in your checking account is not, by our definition, necessarily a flexible decision. Rather, it is this decision that increases our subsequent flexibility for the decision of what we purchase by check. On the other hand, the irreversible commitment of a decision variable to a specific alternative eliminates the flexibility associated with that variable. Prior to zeroing out one's checking account, the decision to buy either a set of golf clubs or a new dishwasher is still flexible. However, the minute we hand over the check in the sporting goods store that decision becomes inflexible.

Chapter 2

Chapter 2 is concerned with placing an economic value on maintaining a given amount of decision flexibility. The value of flexibility, however, is strongly dependent upon the information that might be received during the decision process. The more a decision maker expects to learn in the course of a decision, the more it pays to follow flexible decision strategies. Conversely, the more flexible one's decision strategy, the greater the value of information-gathering. Thus, the concepts of value of information and value of flexibility become special cases of the more general concept of the value of information given flexibility.

The value of information given flexibility measures the value to the decision maker, in economic units, of obtaining a given amount of information together with a given amount of decision flexibility. An upper limit to this quantity, the expected value of perfect information given perfect flexibility (EVPIGPF), may be calculated. The method deviates only slightly from the standard decision theory calculation of the expected value of perfect information (EVPI).

The EVPIGPF is similar to, but more complete than, the concept of EVPI. Whereas EVPI measures the value of perfect information under the assumption that all decision variables may be adjusted to utilize the information, the EVPIGPF explicitly states which decision variables may be adjusted in response to what information. In a real system it may be costly or impossible to maintain flexibility on all decisions while awaiting the arrival of some piece of information. By comparing the costs of maintaining flexibility with the EVPIGPF, the decision maker has a method for deciding which decisions ought to be kept flexible and on which it is more profitable to eliminate flexibility.

Chapter 3

Chapter 3 calculates the EVPIGPF for a particular class of decision problems that we call quadratic. Quadratic decisions may be thought of as approximations to a much broader class of problems characterized by decision and uncertain outcome variables that may take on a continuum of values. The exploration identifies the influence of a number of problem parameters on the value of obtaining information with decision flexibility. One of these parameters is the degree of interaction among the problem decision variables. Specifically, if v represents the value of a particular outcome of a decision, the degree of interaction between a decision variable d_i and a decision variable d_j is measured by the second partial derivative $\partial^2 v/\partial d_i \partial d_j$. This derivative gives the degree to which a change in decision variable d_j influences the effect of a change in decision variable d_i on outcome value.

An interesting result of Chapter 3 states that to a first order approximation the value of obtaining information on one uncertain quantity plus the value of obtaining information on another uncertain quantity will equal the value of obtaining information on both quantities simultaneously, only provided that the information is uncorrelated; that is, provided that learning one quantity does not help us in learning the other. Similarly, the value of obtaining flexibility on one decision plus the value of obtaining flexibility on another decision will, to a first approximation, equal the value of obtaining flexibility jointly, only provided that the two decisions do not interact.

Chapter 4

One of the most common methods for analyzing decision problems is the decision tree. In a decision tree information and flexibility are represented as quantized or discretized approximations by branches emanating from nodes in a tree-like structure. Chapter 4 is concerned with the effect of such quantization on the value of information given flexibility. A method is presented for determining the precise economic loss to be expected from using a quantized rather than an exact information reporting system. In addition, the possibility of designing optimal quantizing systems is demonstrated.

Once again, an analysis is conducted on the special class of quadratic problems. For the two-variable quadratic problem, which contains a single decision and a single uncertain outcome, the expected economic loss from information quantization or flexibility discretization is investigated for several different probability distributions describing the uncertain outcome. Expected loss is found to be relatively insensitive to the particular quantizing or discretizing system chosen. Invariably, the expected economic loss falls quickly with the number of quantizing or discretizing levels employed: roughly 60% of the value of information or 60% of the value of flexibility may be obtained using only two levels, and 80% of the value may be expected using three levels. The implication is that information and flexibility are well represented in a decision tree by as few as three branches from respective state or decision nodes.

Chapter 5

Chapter 5 incorporates the flexibility concept into a useful technique for decision model design and analysis called "sensitivity analysis." One form of sensitivity analysis is proximal analysis. Proximal analysis assumes that the decision problem is approximately quadratic. Thus, the thesis results concerning the quadratic decision problem are directly applicable here. The main results of Chapter 5 provide techniques for more easily estimating the EVPIGPF under the assumptions of the proximal model.

CHAPTER 1

THE CONCEPT OF FLEXIBILITY

Nearly everyone is familiar with the story of a plan that went wrong because it failed to adjust for some unforeseen circumstance. We might say that such a plan lacked flexibility, that a good decision strategy is a flexible one. However, what exactly do we mean by decision flexibility?

1.1 LITERATURE

If one turns to micro-economic literature on the theory of the firm, one will uncover a number of definitions for decision flexibility. For most authors flexibility is, roughly speaking, a property of a decision which makes it easily altered. Hart [6] and Theil [14] define flexible decision rules as dynamic, inflexible decision rules as static. From their point of view if a production plant is flexible, it is possible to diverge from planned values at a date subsequent to their acceptance. Hence, flexibility refers to the ability to modify plans over time. Marschak and Nelson [10] are more specific. For them flexibility is a property of an initial decision which makes subsequent actions less costly or preserves more choices. Stigler [13] and Baumol [1] refer to flexibility in the context of static decision-making; for them it means the rate of change of marginal cost. The smaller a plant's second derivative of total cost, the more flexibility it has. Tisdell [15] illustrates the importance of distinguishing among the various definitions of flexibility by demonstrating that they may have conflicting consequences for decision-making.

In each case the meanings authors have ascribed to the term flexibility have been appropriate only within the relatively narrow contexts of their particular problems. The objective of Chapter 1 is to present a mathematically precise definition which is applicable to a much broader class of problems.

1.2 DECISION ANALYSIS

For our definition and subsequent exploration of flexibility we shall rely upon the theory of decision analysis [7,9]. Decision analysis is a practical discipline combining the techniques of decision theory with the mathematical methods of systems analysis. Decision theory is a logical means for formulating decision strategies under uncertainty.

We define a decision action to be a commitment of scarce resources that is irrevocable in the sense that it cannot be subsequently altered without incurring a nontrivial penalty. A decision strategy is defined as a deliberate selection of a specific decision action from the set of alternative actions. If this choice of action is made without knowing the precise consequences of selecting at least one of the alternative actions, we face a decision under uncertainty.

Both the techniques of decision theory and the concept of flexibility will be made clearer by the analysis of an example.

Example: The Sofa Sale

A young housewife named Linda thought she had found the perfect sofa for the apartment, and at one-half the usual price. "Now if only Jim likes it," she thought, and she couldn't wait to show it to her husband. "I'm sorry lady," apologized the salesman, "that model has been selling like hotcakes. We've only got one left. I can't hold it 'til Saturday for you without a $25 deposit, store rules." "What should I do?" Linda thought. "I'm almost sure Jim will like it, but I certainly can't buy a sofa without his seeing it first." Just as she was pondering her situation she overheard an excited conversation between two elderly women, "Oh, look at that lovely sofa! Wouldn't that be just perfect for our den?" "Why yes, Martha, I believe you're right!"

The only way that Linda can retain flexibility on her decision whether or not to buy the sofa is to make the $25 deposit. Let's imagine that she decides to analyze the problem using some techniques of decision analysis. She observes that there are three possible outcomes to her problem. If she makes the deposit, she and her husband could decide (a) to purchase the sofa, or (b) not to purchase it. If she does not make the deposit, she reasons, it will certainly be sold before Saturday. Let us assume that Linda is able to rank preferentially these three possible outcomes and express this ranking in a numerical fashion. In other words, we assume that she defines a utility function over the set of possible outcomes. Figure 1.1 shows the specific utilities U for this example.

While Linda has control over the decision of whether or not to make the deposit, she does not (for the purposes of this example, at least) have control over whether or not her husband will want to purchase the sofa. The decision whether or not to make the deposit and the ultimate decision of whether or not to purchase the sofa are controllable and, therefore, designated decision variables. Uncontrollable determinants of outcome (in this case the opinion of Linda's husband) are generally called state variables. While the decision maker does not know which state will occur, she does know which state is likely to occur. We assume that she expresses this knowledge in terms of a probability distribution defined over the possible states. In other words, Linda encodes her estimation as to the likelihood of state occurrence in a probability function. In the present example we assume a probability of .8 that the state, "Jim likes sofa," will occur and a probability of .2 that the state, "Jim doesn't like sofa," will occur.

DECISION ALTERNATIVES	STATE VARIABLE OUTCOMES	
	JIM LIKES SOFA	JIM DOESN'T LIKE SOFA
MAKE DEPOSIT	DEPOSIT MADE, SOFA BOUGHT $U = 2$	DEPOSIT MADE, SOFA NOT BOUGHT $U = -0.5$
DON'T MAKE DEPOSIT	DEPOSIT NOT MADE, SOFA NOT BOUGHT $U = 0$	

FIGURE 1.1 POSSIBLE OUTCOMES AND THEIR UTILITIES FOR THE SOFA SALE EXAMPLE

This completes the necessary specification in the problem. The decision maker has specified the alternative actions and uncontrollable states in a mutually exclusive and exhaustive fashion. Subjective probabilities have been assigned to state occurrences, and each conceivable outcome--that is, each possible combination of action and state--is described with an appropriate utility measurement. The decision structure is summarized in "tree" form in Fig. 1.2.

Now, according to the fundamental theorem of decision theory, the best action alternative is the one with the highest expected utility. The expected utility of a given action is defined to be the sum, over all possible states, of the utilities of the state conditioned outcomes multiplied by the probabilities of the corresponding states. The expected utility of the "make deposit" alternative is, therefore,

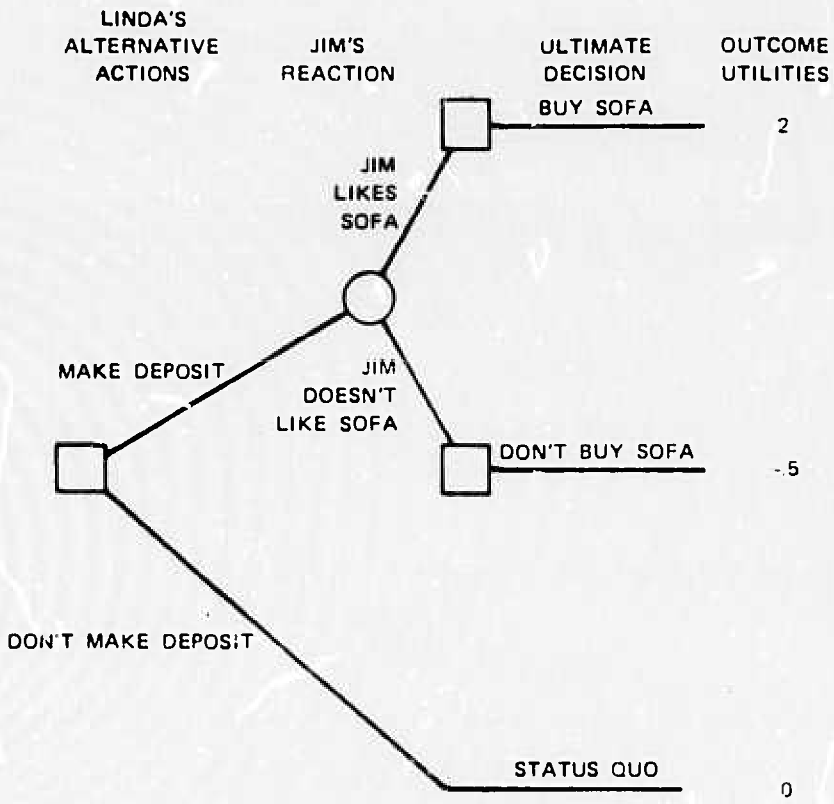


FIGURE 1.2 LINDA'S DECISION TREE

$$
\begin{aligned}
E(\text{"make deposit"}) &= \text{Prob("Jim likes sofa")} \times U(\text{"buy sofa"}) \\
&\quad + \text{Prob("Jim doesn't like sofa")} \\
&\quad \times U(\text{"don't buy sofa"}) \\
&= .8 \times 2 + .2 \times (-.5) = 1.5 \text{ units} . \qquad (1.2.1)
\end{aligned}
$$

Since this exceeds the expected utility of the "don't make deposit" alternative (0 units), the best alternative given these action alternatives, states, probabilities, and utilities of outcomes is to make the deposit. In this case the cost of retaining flexibility on the decision to buy the sofa is well worth the expected gain.

A BASIC DECISION MODEL

The example of this section illustrated some of the essential features of decision analysis. Given an uncertain choice situation, the decision analyst performs a decomposition of the problem, usually following a procedure symbolized by the flow graph shown in Fig. 1.3. In a <u>deterministic phase</u> he specifies the alternative actions, states, and outcomes relevant to the problem and assigns dollar values to the outcomes. In a <u>probabilistic phase</u> he encodes the decision maker's uncertainties on the state variables in a subjective probability distribution and his feelings towards risk in a utility function. Finally, an <u>informational phase</u> may be executed in which calculations of the value of gathering additional information are made. At this point either an optimal action alternative is chosen, or the decision is made to gather further information and the cycle is repeated.

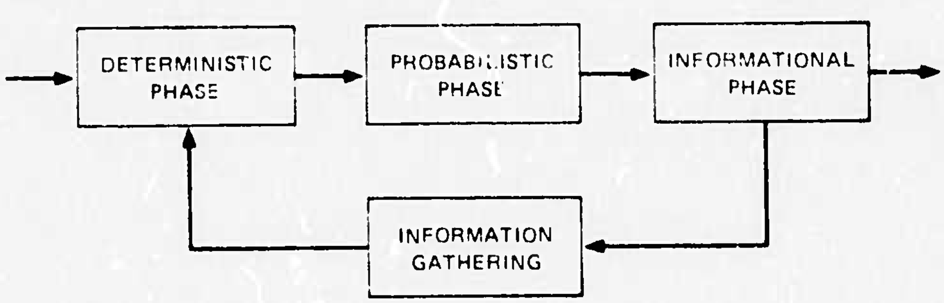


FIGURE 1.3 THE DECISION ANALYSIS CYCLE

A Static Model

If no additional information is anticipated to become available in the course of the decision process, the choice of decision action must be based solely on the decision maker's current state of experience. In this simplest case there are five essential components of the decision maker's resource commitment problem. These are:

1. A feasible set D of mutually exclusive and exhaustive decision action vectors $\underline{d} = (d_1,\ldots,d_m)$. One and only one $\underline{d} \in D$ must be selected.
2. An appropriate set S of mutually exclusive and exhaustive state vectors $\underline{s} = (s_1,\ldots,s_n)$. One and only one $s \in S$ will occur.
3. A probability distribution F defined on S which consistently describes the decision maker's feelings about the likelihood of various states occurring. Usually F is assumed to be independent of the decision $\underline{d}$.
4. A value function $v(\cdot,\cdot)$ defined on the Cartesian product of the sets S and D that describes the decision maker's assessment of the dollar value of each combination of decision alternative and state. Since each outcome is described by a unique combination of decision alternative and state, the value function may be thought of as defining a dollar value for each outcome.
5. A utility $U = u(v)$ defined for each $v(\underline{s},\underline{d})$, $\underline{s} \in S$, $\underline{d} \in D$, which expresses the decision maker's feelings toward risk.

These five model components, denoted $\{D,S,F,v,u\}$, are specified on the basis of the decision maker's existing state of experience, which we explicitly denote by $\mathcal{S}$. One special state of experience is the total knowledge available at the beginning of the problem, the prior experience. When the level of experience is assumed to be the prior experience, we shall use the special symbol $\mathcal{E}$.

According to the principles of decision analysis, we can now locate the decision maker's most preferred action by selecting the $\underline{d} \in D$ which results in the greatest expected utility; that is, the analysis requires us to solve for

$$\underline{d}^* = \max_{\underline{d} \in D}{}^{-1} \int_{\underline{s} \in S} u[v(\underline{s},\underline{d})]\, dF . \qquad (1.2.2)$$

Information Processing

Without doubt, the major contribution of decision theory is the recognition of uncertainty and its partial (or complete in the case of clairvoyance) resolution through information. By information we mean any data which may alter the decision maker's predictions.

If the five facets of the decision model are correctly specified, the decision maker's outcome uncertainty is confined solely to state occurrence. We make this assumption. Stated differently, although S , D , F , v , and u are all based on the decision maker's prior level of experience $\mathcal{E}$, we hypothesize that if $\mathcal{E}$ changes, the effect on the model is completely accounted for by a change in F ; that is, by probability revision. The dependence of the decision maker's probability distribution on his state of experience will be made explicit by using the inferential notation $\{\underline{s}|\mathcal{E}\}$ to denote the probability density function of $\underline{s}$. When we wish to denote explicitly that the expected value of a random vector $\underline{s}$ is based on the state of knowledge, we will use in place of

$$E(\underline{s}) = \int_{\underline{s} \in S} \underline{s} \, dF \tag{1.2.3}$$

the notation

$$\langle \underline{s} | \mathcal{E} \rangle = \int_{\underline{s} \in S} \underline{s} \{\underline{s}|\mathcal{E}\} \; . \tag{1.2.4}$$

If the impact of information is strictly limited to probability revision, the basis for information analysis is provided by Bayes' rule. To illustrate, suppose that some information-gathering method or experiment η produces information or signal y from some set of possible signals Y . Knowledge of y changes the decision maker's distribution on $\underline{s}$ to $\{\underline{s}|y,\mathcal{E}\}$, which is related to the prior distribution $\{\underline{s}|\mathcal{E}\}$ by Bayes' equation,

$$\{\underline{s}|y,\mathcal{E}\} = \frac{\{y|\underline{s},\mathcal{E}\}\,\{\underline{s}|\mathcal{E}\}}{\{y|\mathcal{E}\}} \; . \tag{1.2.5}$$

The quantity $\{y|\underline{s},\mathcal{E}\}$ is the probability of observing a particular

$y \in Y$ for any value of the state vector and is called the likelihood function. The quantity $\{y|\mathcal{E}\}$ is called the preposterior distribution. It is the probability of observing a particular $y \in Y$ before the signal is received and is related to the likelihood function and prior by

$$\{y|\mathcal{E}\} = \int_{\underline{s} \in S} \{y|\underline{s},\mathcal{E}\}\ \{\underline{s}|\mathcal{E}\}\ . \tag{1.2.6}$$

A Dynamic Model

When a decision maker anticipates receiving an information signal y, it is often possible (and usually desirable) to make some components of his strategy contingent upon that signal. In such cases we shall assume that his possible decision strategies are confined to some feasible set $\tilde{D}$ of decision functions defined over Y. Each decision function in $\tilde{D}$ associates a feasible action from D to each possible signal in Y. The optimal strategy is found by determining from the feasible set the function $\underline{d}^*(\cdot)$ that maximizes the expected utility.

$$\underline{d}^*(\cdot) = \max^{-1}_{\underline{d}(\cdot) \in \tilde{D}} \int_{y \in Y} \int_{\underline{s} \in S} u(v[\underline{s},\underline{d}(y)])\ \{\underline{s}|y,\mathcal{E}\}\ \{y|\mathcal{E}\}\ . \tag{1.2.7}$$

An Abstract Representation of the Basic Decision Model

Figure 1.4 is a useful abstract representation of the basic decision model. Problem variables are divided into those that are under the control of the decision maker--decision variables--and those not under his control--state variables. We can visualize the variables as control knobs, the settings of which determine the reading on a utility meter. The decision variables knobs are set by the decision maker. The state variable knobs are set by a disinterested Nature. The objective of the decision maker is to choose the best possible settings for his decision control knobs, those settings which will produce the highest expected reading on the utility meter. If no additional information will be received, possible decision variable settings are the elements of the action set D. However, if a suitable information system exists

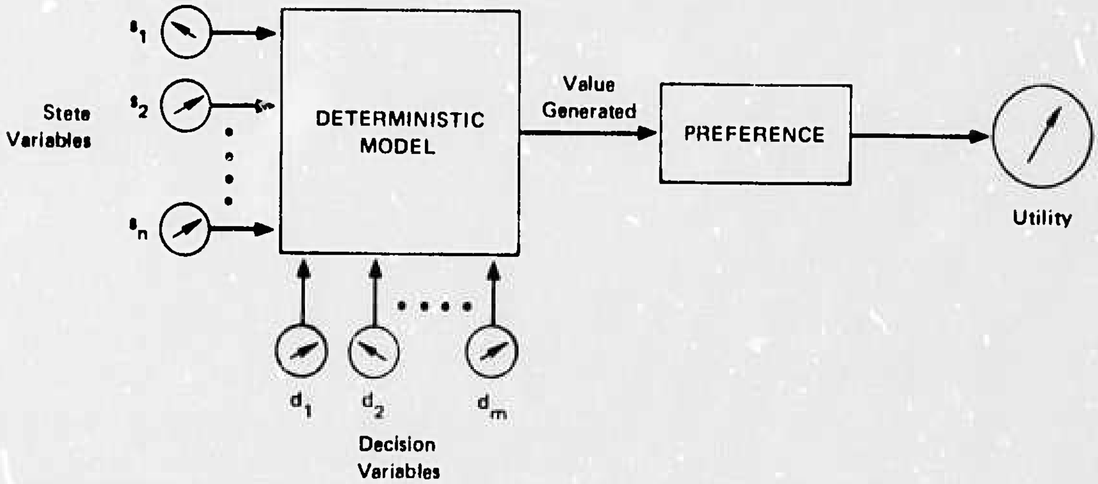


FIGURE 1.4 AN ABSTRACT REPRESENTATION OF THE BASIC DECISION MODEL

and the decision may be conditioned upon the arrival of information signals from this system, then the possible decision variable settings are functions of the signals. In this case a set of feasible decision strategies $\tilde{D}$ is determined, and the decision maker must choose a decision control knob setting (a feasible function of the signal y) from this set.

1.3 A MATHEMATICAL DEFINITION OF FLEXIBILITY

We shall now formulate a mathematical definition of flexibility using the decision analysis framework developed in the last section. The Sofa Sale example illustrated that flexibility may be interpreted as a property of the set of decision alternatives (the choice set) associated with a given decision variable. If the deposit is made, Linda has two alternatives for the purchasing decision--buy or don't buy--subsequent to learning an important piece of information--whether or not her husband likes the sofa. If the deposit is not made, there is only one alternative. Linda will not purchase the sofa. With flexibility, the decision choice set is larger.

We formalize this definition using the decision model illustrated in Fig. 1.5. This figure is identical to that of Fig. 1.4 except that we have illustrated the existence of an information structure η which will yield an information signal y. The structure is assumed

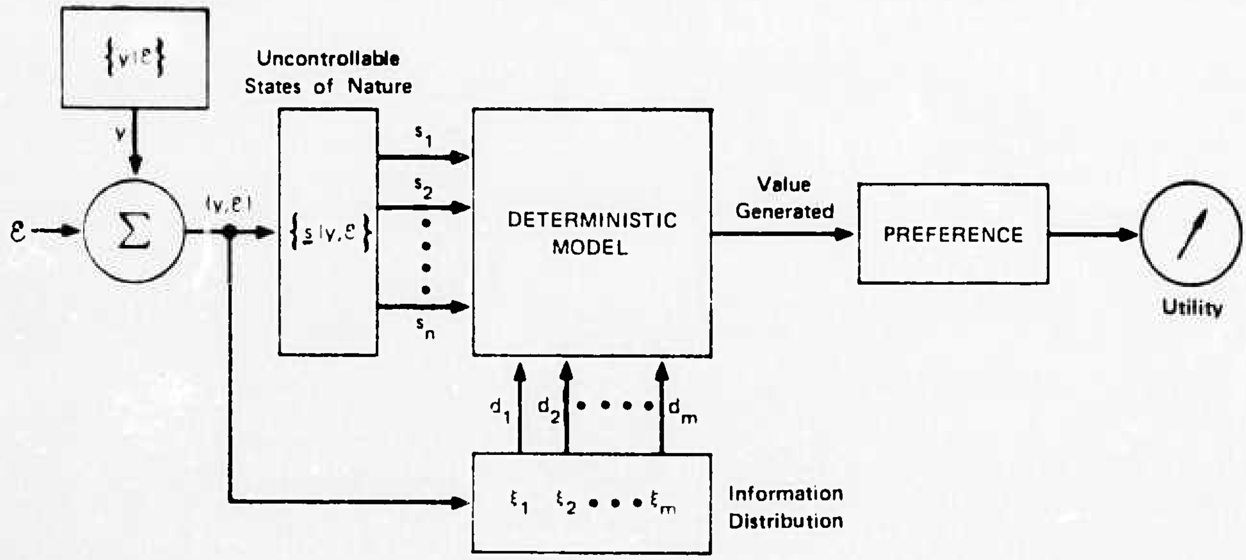


FIGURE 1.5 BASIC DECISION MODEL WITH INFORMATION STRUCTURE

to distribute the signal in such a way that some function of y, denoted $\xi_j(y)$, is available for the setting of the decision variable d_j. A common example is the case in which y represents data received over time and $\xi_j(y)$ is the data received prior to the setting of the j^{th} decision variable. The set of feasible decision functions $\tilde{D}$ will, in general, consist of a set of vector functions $\left\{(d_1,\ldots,d_m)\right\}$ for which the component d_j is a function only of the fraction of total experience $(\xi_j(y),\mathcal{E})$. Imagine now that all decision variables with the exception of d_j have been set to specific decision functions $d_1^o,\ldots, d_{j-1}^o, d_{j+1}^o,\ldots,d_m^o$. We define $\tilde{D}_j^o$ to be the set of possible settings for the decision variable d_j,

$$\tilde{D}_j^o(\mathcal{S}) = \left\{d_j \mid (d_1^o,\ldots,d_{j-1}^o,d_j,d_{j+1}^o,\ldots,d_m^o) \in \tilde{D},\mathcal{S}\right\} . \qquad (1.3.1)$$

We shall define flexibility on the j^{th} decision variable as a property of its decision set $\tilde{D}_j^o$.

DEFINITION 1.3.1: Imagine two decision models $\mathcal{M} = \{D, S, \{\underline{s}|\mathcal{E}\}, v, u\}$ and $\mathcal{M}' = \{D', S, \{\underline{s}|\mathcal{E}\}, v, u\}$ which are identical except possibly for the feasible action sets D and D'. Suppose that there is an information system η and that $\tilde{D}$ and $\tilde{D}'$ are the respective sets of feasible decision strategies which use the information system η. Given the state of experience $\mathcal{S}$ and respective decision strategies for models $\mathcal{M}$ and $\mathcal{M}'$, we say that decision variable d_j is less flexible than d'_j if $\tilde{D}^o_j$ is a proper subset of $\tilde{D}^{o\prime}_j$ (denoted $\tilde{D}^o_j \subset \tilde{D}^{o\prime}_j$).

The decision variable d_j will be said to be flexible or inflexible depending upon whether or not the set $\tilde{D}^o_j$ consists of more than a single (point) element.

Notice that although flexibility is a property of the decision set for d_j, in general, it will depend upon the decision settings for all other decision variables. This corresponds to the common situation in which an action taken now is a decision among later choice sets. Perhaps the action is to acquire a choice set. Perhaps the action is to narrow down an earlier choice set. The decision on whether or not to make the deposit in the sofa-purchasing example can be interpreted in either of these ways. In such cases defining flexibility as the size of the decision choice set is intuitive.

The definition gives us some clues for increasing the flexibility of a decision variable. One method is to choose the setting of $\underline{d}\text{-}d_j$ (which we mean to denote those decision variables in $\underline{d}$ with the exception of d_j) so as to yield a large choice set for d_j. This is what we did in the sofa-purchasing example. Another is to uncover new alternatives so as to increase the size of the set $\tilde{D}$ of feasible decision strategies. Both techniques can be used to analyze the effects of increasing flexibility; the distinction is between an implicit and an explicit evaluation.

1.4 PLANT FLEXIBILITY

In this section we shall apply our definition of relative flexibility to a classical problem in micro-economic theory. The problem has to do with choosing production flexibility in the theory of the firm and is schematically illustrated in Fig. 1.6. A number of various designs for

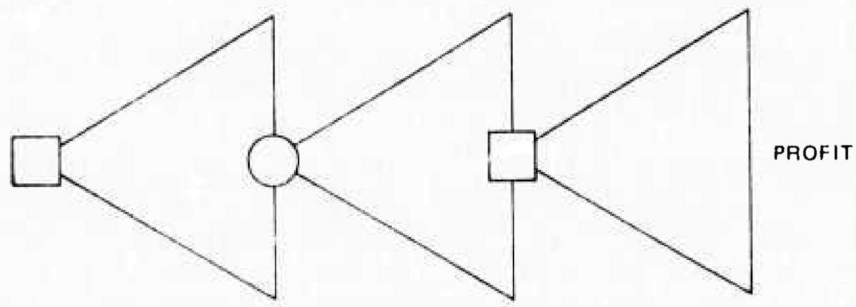


FIGURE 1.6 PLANT FLEXIBILITY DECISION PROBLEM

a production plant are possible. The particular design chosen will determine the output capacity for the plant. At the time plant output capacity is chosen, output price and, therefore, optimal output are not known with precision. However, a probability distribution on output price is known. After the plant has been constructed output price is disclosed, and plant output is set so as to maximize expected profit. It is desired to determine the manner in which varying the plant's output flexibility affects expected profit.

Baumol [1, p. 93] argues that "the existence of uncertainty will lead to the (increased) use of equipment whose scale of operation is flexible." We shall explore this conjecture using two simple models. The first model was devised by Marschak and Nelson [10] and uses Stigler's [13] measure of plant flexibility, a measure which is incompatible with our own. For Stigler, flexibility refers to the rate of change of the plant's marginal cost curve. In the second model, plant flexibility is measured in a manner consistent with Definition 1.3.1.

MODEL 1

Assume the various plant designs are described by quadratic total cost curves,

$$TC = ax^2 + bx + ax_m^2 , \tag{1.4.1}$$

where x is output quantity. Average cost is then

$$AC = ax + b + a\frac{x_m^2}{x} \tag{1.4.2}$$

and

$$AC(x_m) = 2ax_m + b \; , \quad \frac{dAC(x_m)}{dx} = 0 \; , \quad \frac{d^2AC(x_m)}{dx^2} = \frac{2a}{x_m} \; . \tag{1.4.3}$$

Thus, the quantity x_m represents the output with minimum average cost, and the parameter a measures the curvature of the average cost curve (Fig. 1.7). The smaller a is, the flatter the curve at its bottom. Following Marschak and Nelson and Stigler, we shall use $1/a$ as a measure of the plant's flexibility to produce outputs other than that originally planned. According to Definition 1.3.1, however, all such plants would be termed equally flexible since, regardless of a, any output along the positive real axis is feasible.

For price P, profit is given by

$$\pi = Px - ax^2 - bx - ax_m^2 \; . \tag{1.4.4}$$

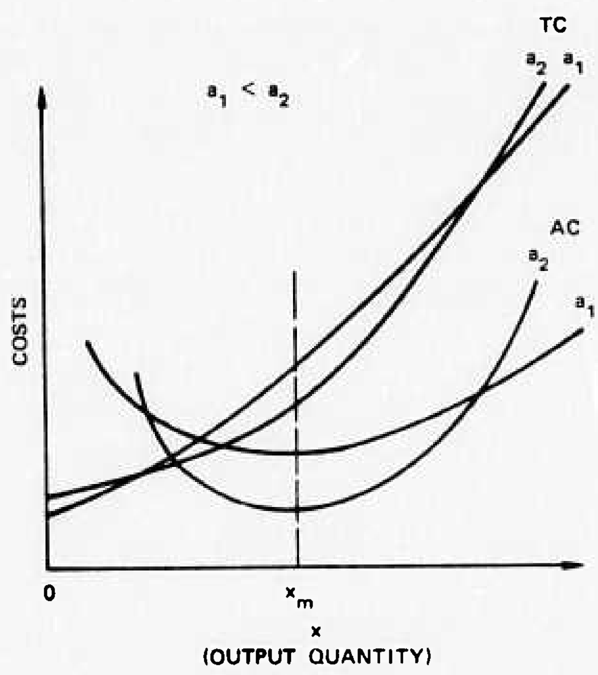


FIGURE 1.7 TOTAL AND AVERAGE COST CURVES FOR PLANT FLEXIBILITY MODEL 1

Assuming $P > b$, the profit maximizing output

$$x^* = \frac{P - b}{2a} \tag{1.4.5}$$

produces an optimal profit

$$\pi^* = \frac{(P - b)^2}{4a} - ax_m^2 . \tag{1.4.6}$$

Now, suppose price is distributed with mean $\bar{P}$ and variance σ^2. Expected profit is then given by

$$E(\pi^*) = \frac{(\bar{P} - b)^2}{4a} - ax_m^2 + \frac{\sigma^2}{4a} . \tag{1.4.7}$$

On obtaining this result, Marschak and Nelson make the observation that as subjective uncertainty increases (as measured by σ^2), expected profit for the more flexible plant (small a) rises relative to expected profit for the less flexible plant (large a). Broadly interpreted, this supports Baumol's claim.

Yet, we are still left with the question of whether $1/a$ is a good measure of production flexibility. One disturbing feature of this measure is that in the absence of uncertainty $(\sigma^2 = 0)$ expected profit still increases as a decreases. If price is known with certainty, should not the potential for adjusting to different prices have zero value?

MODEL 2

This time we assume the various plant designs are described by total cost curve of the form

$$TC = \begin{cases} ax^2 + bx + ax_m^2 , & x \le \bar{x} + A \\ \infty , & x > \bar{x} + A \end{cases} . \tag{1.4.8}$$

The maximum output which can be produced from such plants is $\bar{x} + A$, and, therefore, the magnitude of A determines the size of the plant output choice set. Following Definition 1.3.1, we may take A to be a measure of the plant's output flexibility--the larger A is, the greater the plant's flexibility over its output setting.

Average cost is given by

$$AC = \begin{cases} ax + b + a\dfrac{x_m^2}{x} , & x \le \bar{x} + A \\ \infty , & x > \bar{x} + A \end{cases} . \qquad (1.4.9)$$

Once again, x_m represents the output with minimum average cost (Fig. 1.8). For outputs less than $\bar{x} + A$, profit maximizing output and optimal profit are given by expressions identical to those of Model 1:

$$x^* = \frac{P - b}{2a} , \qquad (1.4.10)$$

$$\pi_A^* = \frac{(P - b)^2}{4a} - ax_m^2 , \quad b \le P \le 2a(\bar{x} + A) + b . \qquad (1.4.11)$$

If price exceeds $2a(\bar{x} + A) + b$, the maximum output $\bar{x} + A$ will be chosen and profit will be given by

$$\pi_A^* = (P - b)(\bar{x} + A) - a(\bar{x} + A)^2 - ax_m^2 , \quad P > 2a(\bar{x} + A) + b . \qquad (1.4.12)$$

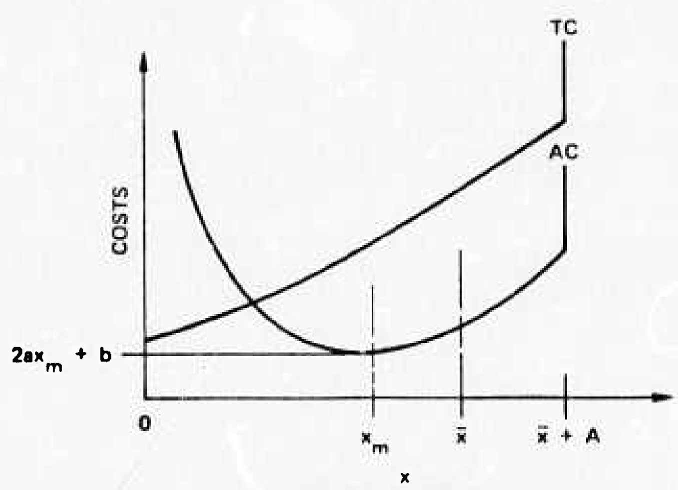


FIGURE 1.8 TOTAL AND AVERAGE COST CURVES FOR PLANT FLEXIBILITY MODEL 2

Suppose price P is normally distributed with standard deviation σ and mean $\bar{P} = 2a\bar{x} + b$. (Actually we should have P strictly greater than b to insure that the optimal output is positive. We assume that the probability of obtaining a $P \le b$ with the normal distribution is small enough to have a negligible effect on the analysis.) The quantity $\bar{x}$ represents the expected plant output, and A is the amount by which maximum plant capacity exceeds the expected operating output. Since minimum average cost is $2ax_m + b$, if $\bar{x}$ exceeds x_m the plant may be expected to make a profit. Denoting by $n(x;m,\sigma)$ the normal probability density with dummy variable x, mean m, and standard deviation σ, the expected profit lost by not choosing a completely flexible plant is

$$\begin{aligned} E(\pi^*_\infty - \pi^*_A) &= \int_{2a(\bar{x}+A)+b}^{\infty} (\pi^*_\infty - \pi^*_A)\, n(P;2a\bar{x} + b,\sigma)\, dP \\ &= \int_{2a(\bar{x}+A)+b}^{\infty} \frac{[P-2a(\bar{x} + A)- b]^2}{4a}\, n(P;2a\bar{x} + b,\sigma)\, dP \\ &= \frac{\sigma^2}{4a} \int_{2aA/\sigma}^{\infty} (y - \frac{2aA}{\sigma})^2\, n(y;0,1)\, dy\,, \end{aligned} \qquad (1.4.13)$$

where we have used the change of variable $y = (P-2a\bar{x}-b)/\sigma$. As an aid to evaluation, the integral

$$I(\frac{2aA}{\sigma}) = \int_{2aA/\sigma}^{\infty} (y - \frac{2aA}{\sigma})^2\, n(y;0,1)\, dy \qquad (1.4.14)$$

has been plotted in Fig. 1.9. Observe that the difference in expected profit between the completely flexible plant $(A = \infty)$ and the plant with minimum flexibility $(A = 0)$ is $\sigma^2/8a$. Since expected profit for the completely flexible plant is $[(\bar{P}-b)^2]/4a - ax_m^2 + \sigma^2/4a$, we have

$$E(\pi^*_A) = \frac{\sigma^2}{4a}[1 - I(\frac{2aA}{\sigma})] + a(\bar{x}^2 - x_m^2)\,. \qquad (1.4.15)$$

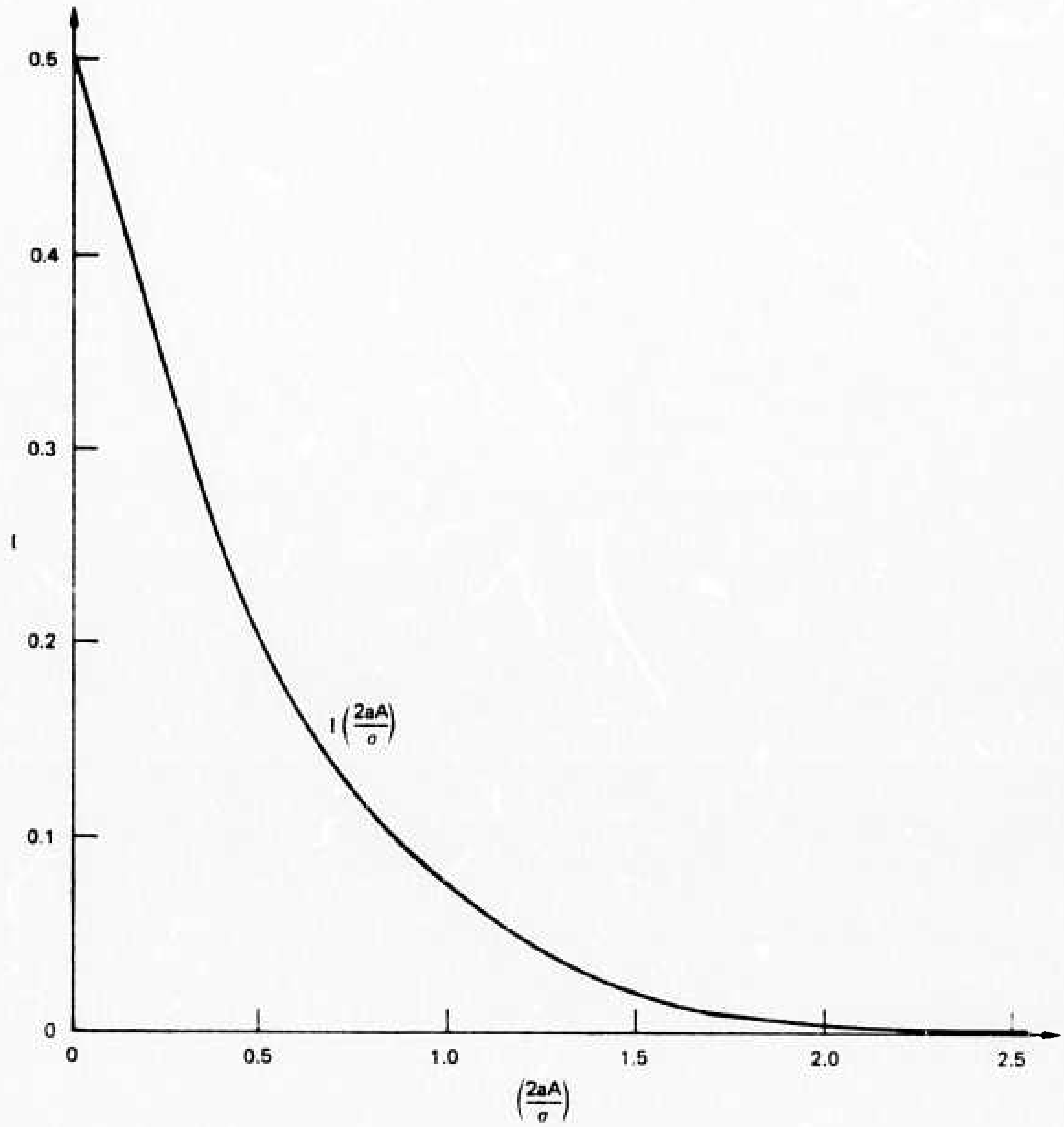


FIGURE 1.9 PLOT OF $I\left(\frac{2aA}{\sigma}\right)$

Finally, observe that

$$\frac{\partial E(\pi_A^*)}{\partial A} = \frac{\sigma}{2} \cdot [-I'(\frac{2aA}{\sigma})] > 0 \ , \qquad (1.4.16)$$

I' denoting the derivative of I. Each of the two factors in (1.4.16) increases with σ. Thus, the gain in expected profit achieved through flexibility is larger, the greater the price uncertainty as measured by σ. If there is no uncertainty $(\sigma = 0)$, there is no advantage in having flexibility; but, as σ increases, expected profit for a more flexible plant rises relative to expected profit for a less flexible plant. This model not only supports Baumol's claim, it also conforms well with our intuitive notions of how the value of flexibility should depend on uncertainty.

CONCLUSIONS

Our definition seems to perform reasonably well when applied to the problem of determination of plant flexibility. Notably, it leads to results which conform with our intuitive feelings. However, unlike Stigler's definition, ours says nothing about the relative costs of varying a decision variable over a choice set. While this may seem to be a serious limitation, we shall see that there is an important application to problems in which we ignore the costs of varying pre-set decision variables. This topic is the subject of Chapter 2 of the thesis.

CHAPTER 2

THE VALUE OF INFORMATION GIVEN FLEXIBILITY

Everything else being equal, we would all agree that the more flexible of two plans is the more desirable. Everything else, however, is usually not equal. A more flexible strategy is typically more inconvenient and often more costly to follow. Therefore, an economic measure of the value of flexible over inflexible plans ought to be quite useful to the decision maker. In this chapter we seek such an economic measure.

The example in the latter part of Chapter 1 has already demonstrated a simple and fundamental rule concerning the value of flexibility: The more you expect to learn in the course of a decision (the more uncertainty will be resolved), the more it pays to follow flexible decision strategies. The value of flexibility is thus dependent upon expected information. However, we can turn the argument around: The more flexible your decision strategy, the greater the value of information-gathering. In terms of value, the concepts of information and flexibility are inseparable. If you have no flexibility in your decisions, information has no value. If you expect to receive no information, there is no value to having flexible decision rules.

Therefore, what we really seek is the value of a given amount of information and flexibility. We are in luck. With just a little ingenuity we can turn the well-known expected value of information calculation from decision theory into a calculation that will tell us the economic worth of any combination of information and flexibility.

2.1 PERFECT INFORMATION GIVEN PERFECT FLEXIBILITY

The simplest case, which we now present, will demonstrate the required logic for economically evaluating information with flexibility.

Suppose that the decision model is the basic decision model of Fig. 1.4 and that the time order of the decision process is as illustrated in Fig. 2.1. The decision maker must choose the action vector $\underline{d}$ so as to obtain a value lottery with the highest utility. The desired value of $\underline{d}$, denoted $\underline{d}^*$, will be the solution to

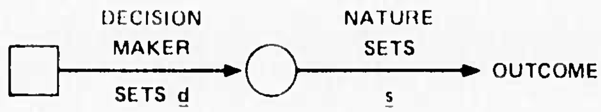


FIGURE 2.1 DECISION PROCESS WHEN NEITHER INFORMATION NOR FLEXIBILITY IS AVAILABLE

$$\underline{d}^* = \max_{\underline{d}\in D}{}^{-1} \langle u|\underline{d},\mathcal{E}\rangle \ . \tag{2.1.1}$$

Equation (2.1.1) is merely a restatement of Eq. (1.2.2) using the inferential notation described in Section 1.2. Use of this decision strategy will result in a maximal expected utility which we will denote by $\langle u|\mathcal{E}\rangle$:

$$\langle u|\mathcal{E}\rangle = \langle u|\underline{d}^*,\mathcal{E}\rangle \ . \tag{2.1.2}$$

Now, suppose that a clairvoyant offers us perfect information on the outcome of state variable s_i , but that we are restricted to using such information only for the setting of decision variable d_j . In other words, at the time the i^{th} state variable is revealed, flexibility exists only on the j^{th} decision variable. Schematically, the time order of the decision process is now as illustrated in Fig. 2.2.

The maximum amount of money the decision maker should be willing to pay to convert the decision problem in Fig. 2.1 to the decision problem in Fig. 2.2 we define to be the *expected value of perfect information on state variable s_i given perfect flexibility on decision variable d_j* .

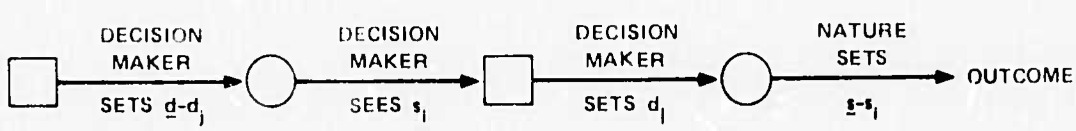


FIGURE 2.2 DECISION PROCESS WITH PERFECT INFORMATION ON s_i GIVEN PERFECT FLEXIBILITY ON d_j

The information is said to be perfect because it totally eliminates uncertainty on the i^{th} state variable. The flexibility is said to be perfect because receipt of the information does not in any way limit the basic alternatives available for the j^{th} decision. The expected value of perfect information on s_i given perfect flexibility on d_j, denoted $\langle v_{Cs_iFd_j}|\mathcal{E}\rangle$, may be obtained from an expected value of perfect information calculation in which only the flexible decision variable is adjusted to compensate for the given information.

CALCULATION PROCEDURE

Let information on s_i given flexibility on d_j be purchased at a price p. Calculation of $\langle v_{Cs_iFd_j}|\mathcal{E}\rangle$ proceeds as follows:

1. Find the optimal decision strategy for decision variable d_j.

$$d_j^*(s_i,\underline{d}-d_j) = \max_{d_j}{}^{-1} \int_{\underline{s}-s_i} \langle u|\underline{s},\underline{d},p,\mathcal{E}\rangle \{\underline{s}|s_i,\mathcal{E}\} . \qquad (2.1.3)$$

2. Solve for the optimal decision settings for the remaining decision variables.

$$(\underline{d}-d_j)^* = \max_{\underline{d}-d_j}{}^{-1} \int_{s_i} \langle u|s_i,d_j^*,\underline{d}-d_j,p,\mathcal{E}\rangle \{s_i|\mathcal{E}\} . \qquad (2.1.4)$$

3. The utility of the lottery with information on s_i, flexibility on d_j is then

$$\langle u|Cs_iFd_j,p,\mathcal{E}\rangle = \langle u|d_j^*,(\underline{d}-d_j)^*,p,\mathcal{E}\rangle . \qquad (2.1.5)$$

4. The value of p which satisfies

$$\langle u|Cs_iFd_j,p,\mathcal{E}\rangle = \langle u|\mathcal{E}\rangle \qquad (2.1.6)$$

is defined as the expected value of perfect information on s_i given perfect flexibility on d_j.

PERFECT INFORMATION ON A SET OF STATE VARIABLES GIVEN PERFECT FLEXIBILITY ON A SET OF DECISION VARIABLES

Perfect information on a subset of state variables, say all s_i, $i \in I \subseteq \{1,\ldots,n\}$, given flexibility on some subset of decision variables $\{d_j\}_{j \in J}$, $J \subseteq \{1,\ldots,m\}$, would be defined as the information structure in which the value of each s_i, $i \in I$ was revealed to the decision maker prior to his setting of the decision variables d_j, $j \in J$. The value of such information given flexibility would be defined and calculated in an obvious way following steps similar to 1-4 above. If the decision maker is allowed to adjust all decision variables in response to information, we say that he has information given complete flexibility. This is the usual assumption for the analysis of information in decision theory literature.

CASE OF LINEAR UTILITY

A special case in which the value of information given flexibility is especially easy to calculate occurs if the utility function is linear in value. In this situation, if Cs_IFd_J denotes the information structure with clairvoyance on state variables s_i, $i \in I$ and flexibility on decision variables d_j, $j \in J$, then the value of this information-flexibility structure is given by

$$\langle v_{Cs_IFd_J}|\mathcal{E}\rangle = \langle v|Cs_IFd_J,\mathcal{E}\rangle - \langle v|\mathcal{E}\rangle . \qquad (2.1.7)$$

Equation (2.1.7) is easily proven. Taking the example of Fig. 2.2 and assuming $u = a + bv$, $b \neq 0$,

$$\langle u|Cs_iFd_j,p,\mathcal{E}\rangle - \langle u|\mathcal{E}\rangle = a + b(\langle v|Cs_iFd_j,\mathcal{E}\rangle - p) - a - b\langle v|\mathcal{E}\rangle$$

$$= b(\langle v|Cs_iFd_j,\mathcal{E}\rangle - \langle v|\mathcal{E}\rangle - p) . \qquad (2.1.8)$$

By definition, when $p = \langle v_{Cs_iFd_j}|\mathcal{E}\rangle$, the left-hand side of (2.1.8) is zero, implying

$$\langle v_{Cs_iFd_j}|\mathcal{E}\rangle = \langle v|Cs_iFd_j,\mathcal{E}\rangle - \langle v|\mathcal{E}\rangle . \qquad (2.1.9)$$

RELATION OF INFORMATION GIVEN FLEXIBILITY TO DEFINITION 1.3.1

Information given flexibility increases flexibility, as defined in Chapter 1, by increasing the size of the decision set $\tilde{D}$. As a result, the decision set for d_j, $\tilde{D}_j$, is larger both for the prior state of experience $\mathcal{E}$ and for the posterior state of experience $(s_i,\mathcal{E})$.

When the state of experience is $\mathcal{E}$ and the model is as illustrated in Fig. 2.1, the feasible decision set consists of all m-dimensional vectors lying in some subset D of the Euclidian space E^m. Using the notation of Chapter 1,

$$D_j^o(\mathcal{E}) = \left\{d_j \,|\, (d_1^o,\ldots,d_{j-1}^o,d_j,d_{j+1}^o,\ldots,d_m^o) \in D,\mathcal{E}\right\}. \qquad (2.1.10)$$

If the model is as illustrated in Fig. 2.2, the j^{th} decision variable is more flexible than (or at least as flexible as) the model of Fig. 2.1 since d_j may now be a function of s_i:

$$D_j^{o'}(\mathcal{E}) = \left\{d_j(\cdot) \,|\, (d_1^o,\ldots,d_{j-1}^o,d_j(s_i),d_{j+1}^o,\ldots,d_m^o) \in D \text{ for all } s_i,\mathcal{E}\right\}. \qquad (2.1.11)$$

Because the set of functions from the space of s_i to E^1 includes the real numbers as a trivial case, $D_j^o(\mathcal{E}) \subseteq D_j^{o'}(\mathcal{E})$.

Once the information s_i becomes known, in the case of Fig. 2.1 the feasible decision set for d_j consists of a single element,

$$D_j^o(s_i,\mathcal{E}) = \left\{d_j^o\right\}. \qquad (2.1.12)$$

However, in the case of Fig. 2.2 it consists of all real numbers that put $\underline{d}$ in D:

$$D_j^{o'}(s_i,\mathcal{E}) = \left\{d_j \,|\, (d_1^o,\ldots,d_{j-1}^o,d_j,d_{j+1}^o,\ldots,d_m^o) \in D,s_i,\mathcal{E}\right\}. \qquad (2.1.13)$$

Hence,

$$D_j^o(s_i,\mathcal{E}) \subseteq D_j^{o'}(s_i,\mathcal{E}) .$$

2.2 APPLICATIONS OF THE EVPIGPF

Just as the expected value of perfect information (EVPI) is a useful yardstick for evaluating information-gathering systems, the expected value of perfect information given perfect flexibility (EVPIGPF) is useful for evaluating information distribution systems.

Suppose a decision maker is considering the construction or purchase of some information-gathering and distribution system. The cost of the proposed system will consist, first of all, of information collection, tabulation, and storage costs. In addition, however, there will be costs involved in having a particular piece of that information available at a specific time or available to a specific individual in the decision process. Different information distribution systems will incur different costs, just as systems which gather different information have differing costs. By placing a value on completely resolving specific uncertainties for specific decisions, the EVPIGPF allows the decision maker to consider seriously only those information-gathering and distribution schemes whose costs do not exceed this value. It may thus be used to evaluate the various information-distribution structures that might be used in a given resource commitment problem.

We can state this use somewhat differently in terms of evaluating flexibility rather than information. Although decision analysis methodology implicitly assumes decisions to be irrevocable, in many cases decision variables can be reset at some cost. As stated by Howard [8, p. 507],

> An executive viewing the results of a decision analysis may think: "It couldn't come out that bad because I would have done something about it." The analysis does not generally take into account the ability to compensate for ultimate state variable changes through adjustments in decision variables.

Flexibility provides a means, and the EVPIGPF provides a measure of the value of taking into account the possibility of compensating for ultimate state variable changes.

EXAMPLE: THE ENTREPRENEUR'S PRICE QUANTITY DECISION

An entrepreneur must decide upon a price and quantity for his product. He is uncertain about the total cost c per item but feels that it may be represented by the uniform distribution of Fig. 2.3. He knows that the demand for his product will be a decreasing function of his price, but for any given price he is uncertain as to the exact quantity of his product demanded. For this reason he hypothesizes the following functional form for demand x :

$$x = \frac{a}{p} - b - e \ , \tag{2.2.1}$$

where

x = demand (in thousands of units),

p = price (in thousands of dollars),

a,b = parameters of the demand curve, and

e = a random variable independent of c and uniformly distributed from zero to one.

Figure 2.4 shows the probability density for e and the demand curve $x(p)$. Further let

q = quantity produced (in thousands of units) and

v = net profit (in millions of dollars).

We wish to determine our entrepreneur's expected net profit and the value to him of using numerous perfect information-perfect flexibility structures. In other words, we would like to know how much it is worth

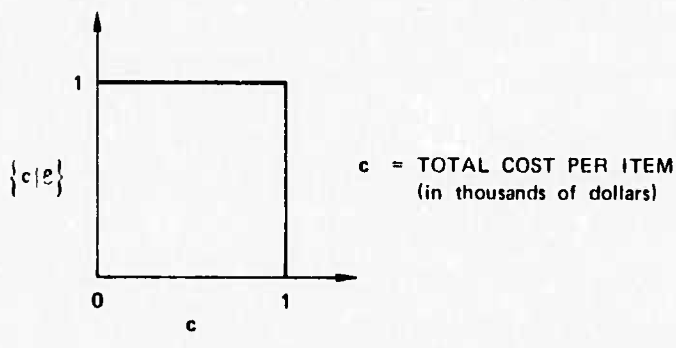


FIGURE 2.3 PROBABILITY DENSITY FUNCTION FOR PRODUCTION COST IN THE ENTREPRENEUR'S DECISION

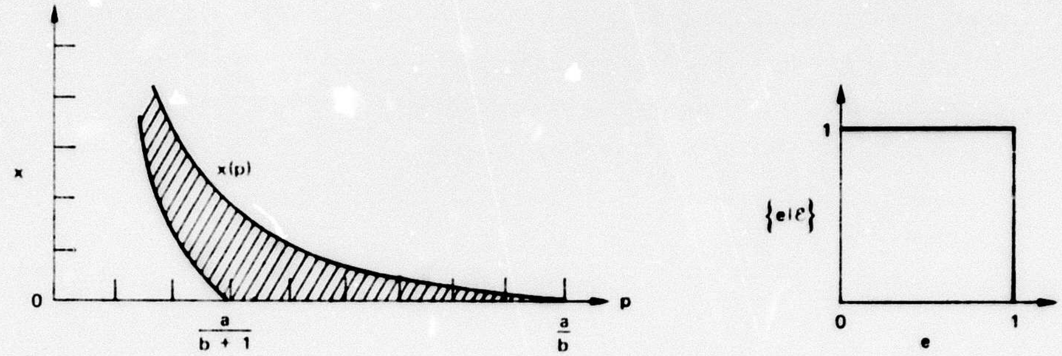


FIGURE 2.4 THE DEMAND CURVE AND THE PROBABILITY DENSITY FUNCTION FOR THE DEMAND PARAMETER e IN THE ENTREPRENEUR'S DECISION

to the entrepreneur to obtain perfect information on various state variables if he uses that information when setting various decision variables. In all there are $3 \times 3 = 9$ possible information-flexibility combinations (excluding the null structure).

The computations are performed in Appendix B and summarized in Table 2.1 for particular parameter values of $a = 2.25$ and $b = .5$. Observe first of all that the expected value of the entrepreneurial venture is half a million dollars and is obtained through an optimal decision strategy of setting price at $1,000 and quantity at 1,250 units. The entries in the table-- V_{CF}, p^*, and q^* -- respectively denote the value of the information-flexibility structure and the optimal decision strategy appropriate to the structure corresponding to a given location in the table. For example, the value to the decision maker of obtaining clairvoyance on the demand parameter e for the purpose of setting his production quantity q is $128,680. It may be possible for our entrepreneur to pick a price, estimate the demand parameter e through a market survey or trial marketing venture, and then set his quantity. According to the table, if he does this he should set price at 1.061 thousand dollars, conduct his estimation of e, and then set quantity at $1.621 - e$ thousands of units. Further, this action should only be considered if a good estimate of e can be obtained for under $128,000.

TABLE 2.1

Value of and Decision Strategy for Various Information-Flexibility Structures in the Entrepreneur's Decision

Information About \ Flexibility On	p	q	p and q
c	$V_{CF} = \$0$ $p^* = 1.0$ $q^* = 1.25$	$V_{CF} = \$42,411$ $p^* = .964$ $q^* = \begin{cases} 0 & \text{if } c > .964 \\ 1.833-1.037c & \text{otherwise} \end{cases}$	$V_{CF} = \$139,416$ $p^* = \sqrt{\frac{(.45-c)c}{2}}$ $q^* = \frac{2.25-c}{\sqrt{\frac{(.45-c)c}{2}}} - .5$
e	$V_{CF} = \$151,639$ $p^* = \frac{2.25}{1.595+e}$ $q^* = 1.095$	$V_{CF} = \$128,680$ $p^* = 1.061$ $q^* = 1.621-e$	$V_{CF} = \$151,924$ $p^* = \sqrt{\frac{2.25}{1+2e}}$ $q^* = \sqrt{.45(.5+e)} - .5-e$
c and e	$V_{CF} = \$151,639$ $p^* = \frac{2.25}{1.595+e}$ $q^* = 1.095$	$V_{CF} = \$128,680$ $p^* = 1.061$ $q^* = 1.621-e$	$V_{CF} = \$271,915$ $p^* = \sqrt{\frac{2.25c}{.5+e}}$ $q^* = \sqrt{\frac{2.25(.5+e)}{c}} - .5-e$

$\langle v|e\rangle = .5 = \$500,000$

$p^* = 1 = \$1,000$

$q^* = 1.25 = 1,250$ units

Knowledge of EVPIGPF's can generate insight that is not provided by EVPI's alone. For example, observe that the value of clairvoyance on costs given flexibility on price is zero, but the value of clairvoyance on costs given flexibility on price and quantity is $139,416. Information about costs is useful for setting price but only if that information is used for setting quantity as well. Once quantity has been fixed, price must be set so as to clear the inventory and costs are no longer a consideration. Insight may also be provided on decision timing. If information on the demand parameter e is purchased, virtually all the usefulness of the information, $151,639 worth, can be obtained using it only to set price Delaying production until after this information becomes available will only be worth an additional $285!

2.3 INFORMATION GIVEN PARTIAL FLEXIBILITY

We referred to the above calculations as situations of "perfect information with perfect flexibility." This is because a clairvoyant was imagined to have provided us with perfect information on the outcome of state variables, and we presumed that decision variables could be set anywhere within the feasible action set. In many cases the approximation of the clairvoyant is not satisfactory for the evaluation of an information-gathering scheme; we must consider the purchasing of an information signal or experiment rather than perfect information. This presents little conceptual difficulty, however, as Bayes' equation allows us to calculate the effect of experimentation on our prior distribution. Hence, we can evaluate the economic impact on our profit lottery.

Similarly, the delaying of a decision so that more information may be gathered may well restrict the range over which that decision variable may be varied. If our entrepreneur were living in our present day of shortages and potential government legislated price freezes, he might find that a short delay could seriously restrict his range of feasible output quantities and prices.

The model appropriate for representing diminished flexibility depends on the nature of the problem being investigated. In some cases, flexibility for a particular decision variable is itself a decision under the control of the decision maker. Contracts may be negotiated which spell out in legal terms the latitude available for some action. A specific example would be the purchase of an "option to buy" so many units of some commodity. In other situations, the restriction of flexibility is the work of Nature and under little control of the decision maker.

A MODEL FOR PARTIAL FLEXIBILITY

A fairly general model for evaluating information and flexibility is the following. Imagine that for a price p, instead of having to set $\underline{d}$ prior to learning $\underline{s}$, our decision maker takes some action f which is designed to preserve a measure of flexibility on the j^{th} decision variable d_j. The flexibility is preserved until after an information system η can be made to produce a signal y from some set of possible signals Y.

The precise flexibility maintained for d_j will, in general, depend upon both $\underline{d}-d_j$ and $\underline{s}$. Let $D_j^f(\underline{d}-d_j,\underline{s})$ denote the choice set for d_j faced by the decision maker if the flexibility producing action is taken. The primary decision problem and the decision problem with the information system and partial flexibility are illustrated in Fig. 2.5.

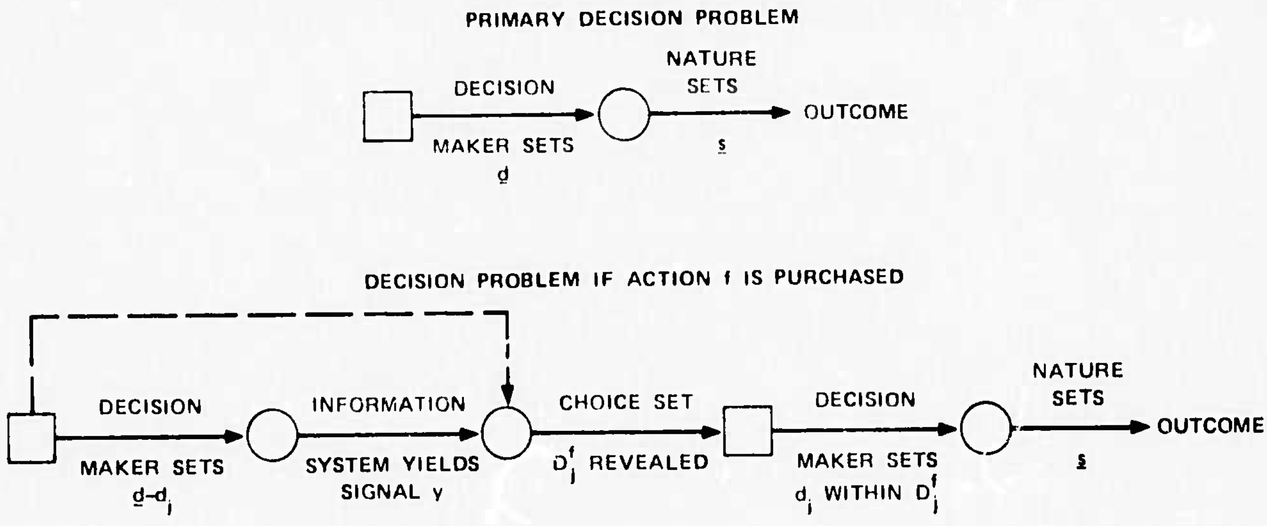


FIGURE 2.5 PRIMARY DECISION PROBLEM AND DECISION PROBLEM WITH INFORMATION SYSTEM AND PARTIAL FLEXIBILITY ON d_j

Value of Information Given Partial Flexibility

The value of the information system η with the partial flexibility on d_j is defined as the maximum price p our decision maker should be willing to spend to achieve the problem conversion shown in Fig. 2.5.

Calculation of this value for the general model described above will, in most cases, be quite difficult, so a number of additional assumptions are likely to be made. We shall illustrate the calculation under two additional assumptions:

1. The choice set D_j^f is uncertain but independent of $\underline{d}-d_j$.
2. Choice set uncertainty is limited so that D_j^f must be one of the K sets $D_j^f(k)$, $k = 1,\ldots,K$. The probability that set $D_j^f(k)$ results given that the information signal is y is $p_k(y)$.

For notational convenience we shall also assume that any information on $\underline{s}$ that our decision maker is able to deduce from the outcome $D_j^f(k)$ has been incorporated into the signal y . (Following the usual assumption of information analysis we assume that the data y is independent of the actions $\underline{d}$.)

To begin the calculation, assume that the choice set is $D_j^f(k)$ and the signal is y . The optimal feasible action for d_j will be the solution to

$$\langle u|\underline{d}-d_j,y,D_j^f(k),p,\mathcal{E}\rangle = \max_{d_j \in D_j^f(k)} \int_{\underline{s}} \langle u|\underline{d},\underline{s},p,\mathcal{E}\rangle \{\underline{s}|y,\mathcal{E}\} , \qquad (2.3.1)$$

where the posterior distribution $\{\underline{s}|y,\mathcal{E}\}$ is given by Bayes' equation (1.1.5). Now, $p_k(y)$ is the probability that the choice set will be $D_j^f(k)$ given that the signal is y , and $\{y|\mathcal{E}\}$ is the preposterior probability that the data y will actually be obtained. Therefore, the expected utility of the optimal decision strategy, given the information gathering technique η and partial flexibility on d_j purchased at a price p , is

$$\langle u \mid \eta, f, p, \mathcal{E} \rangle = \max_{\underline{d}-d_j} \int_{y \in Y} \sum_{k=1}^{K} \langle u \mid \underline{d}-d_j, y, D_j^f(k), p, \mathcal{E} \rangle \, p_k(y) \, \{y \mid \mathcal{E}\} \, . \qquad (2.3.2)$$

Finally, the value of p that satisfies

$$\langle u \mid \eta, f, p, \mathcal{E} \rangle = \langle u \mid \mathcal{E} \rangle \qquad (2.3.3)$$

is the value of the information system η with the partial flexibility on d_j produced by action f .

CHAPTER 3

DECISION MODELS WITH QUADRATIC VALUE FUNCTIONS

In this chapter we shall exploit the special characteristics of a quadratic function in an effort to gain a better understanding of the value of obtaining information given flexibility. The analysis will be built around the basic decision model of Fig. 1.4 for the special case in which (1) the decision maker's value function is quadratic in both state and decision variables, and (2) the decision maker's utility function is linear in wealth.

Almost all of the essential features of the quadratic problem are present in the case in which there are two state variables and two decision variables. Consequently, we begin the analysis with the following example.

3.1 EXAMPLE: A FOUR-VARIABLE PRODUCTION PROBLEM

A firm produces two outputs using fixed proportions of two inputs. Each unit of output one requires a_1 units of input one, a_2 units of input two. Similarly, each unit of output two requires b_1 units of input one and b_2 units of input two. The marginal revenue produced by selling the two outputs smoothly diminishes as the quantity of either output increases ("diminishing marginal returns"). Specifically, we assume that the revenue is a quadratic function of the respective output quantities x_1 and x_2 :

$$R = -x_1^2 - x_2^2 + 2qx_1x_2 + w_1x_1 + w_2x_2 + h \ . \qquad (3.1.1)$$

Output units have been chosen so as to make the coefficients of the quadratic terms -1 . The quantity q measures the complementarity (-q the substitutability) between the two output products.

Let the respective per unit prices of the inputs be

$$p_1 = m_1 + s_1 \quad \text{and} \quad p_2 = m_2 + s_2 \ , \qquad (3.1.2)$$

where m_1 and m_2 are mean prices and s_1 and s_2 are random variables with zero expectations. The net profit to the firm will be (see Fig. 3.1)

$$\pi = -x_1^2 - x_2^2 + 2qx_1x_2 + w_1x_1 + w_2x_2 + h$$
$$- x_1(a_1p_1 + a_2p_2) - x_2(b_1p_1 + b_2p_2) \quad . \quad (3.1.3)$$

Our objective is to determine the quantities x_1 and x_2 that will maximize expected profit. Let us measure these quantities from the values $\tilde{x}_1$ and $\tilde{x}_2$ that would be optimal if the input prices are at their mean values. Since $\partial^2\pi/\partial x_1^2 = \partial^2\pi/\partial x_2^2 = -2$, $\partial^2\pi/\partial x_1\partial x_2 = 2q$, the concavity condition

$$\frac{\partial^2\pi}{\partial x_1^2} < 0 \qquad \begin{vmatrix} \dfrac{\partial^2\pi}{\partial x_1^2} & \dfrac{\partial^2\pi}{\partial x_1\partial x_2} \\[2ex] \dfrac{\partial^2\pi}{\partial x_1\partial x_2} & \dfrac{\partial^2\pi}{\partial x_2^2} \end{vmatrix} > 0 \qquad (3.1.4)$$

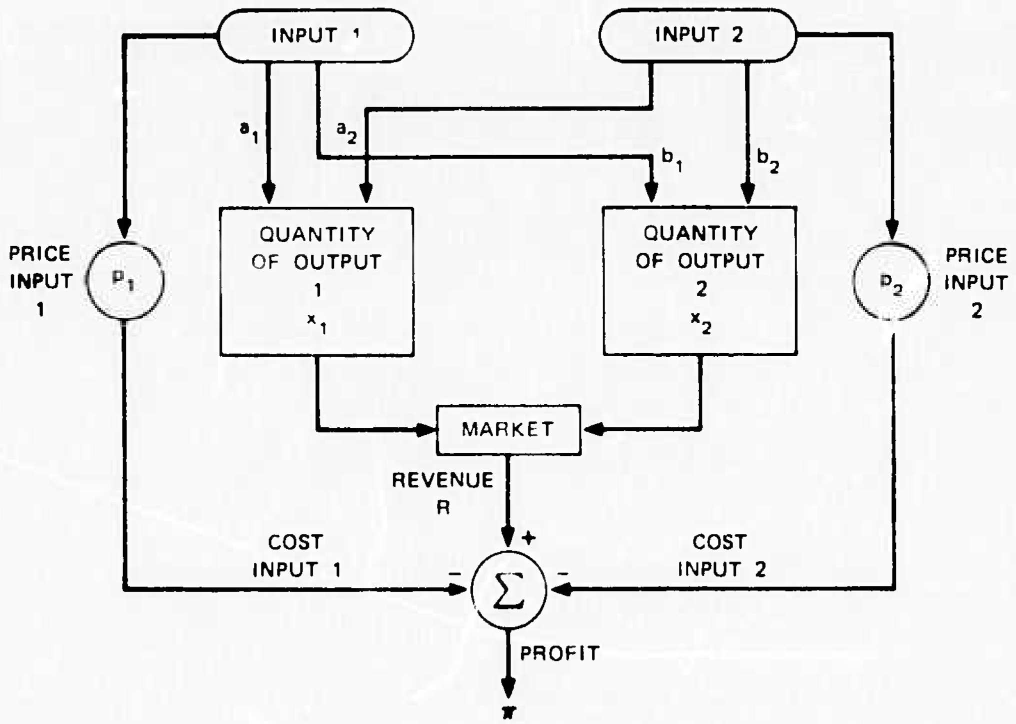


FIGURE 3.1 A FOUR-VARIABLE PRODUCTION PROBLEM

is satisfied for $|q| < 1$. This assumption insures that for every combination of p_1 and p_2, π has a unique maximum. Putting prices at their mean values, differentiating, and setting the results to zero, we obtain

$$\tilde{x}_1 = \frac{w_1 - a_1m_1 - a_2m_2 + q(w_2 - b_1m_1 - b_2m_2)}{2(1 - q^2)} ,$$

$$\tilde{x}_2 = \frac{q(w_1 - a_1m_1 - a_2m_2) + w_2 - b_1m_1 - b_2m_2}{2(1 - q^2)} . \tag{3.1.5}$$

Defining d_1 and d_2 as deviations from these values,

$$x_1 = d_1 + \tilde{x}_1 \quad \text{and} \quad x_2 = d_2 + \tilde{x}_2 , \tag{3.1.6}$$

profit, in terms of the deviations d_1 and d_2 of the output levels from their deterministic optimums and in terms of the deviations s_1 and s_2 of the input prices from their mean values, is given by

$$\bar{\pi}(s_1,s_2;d_1,d_2) = -d_1^2 - d_2^2 + 2qd_1d_2 - d_1(a_1s_1 + a_2s_2) - d_2(b_1s_1 + b_2s_2)$$

$$- \tilde{x}_1(a_1s_1 + a_2s_2) - \tilde{x}_2(b_1s_1 + b_2s_2) + \tilde{\pi} , \tag{3.1.7}$$

where

$$\tilde{\pi} = \pi(\tilde{x}_1,\tilde{x}_2;m_1,m_2) \tag{3.1.8}$$

is the maximum profit if input prices are set at their mean levels. Since the last three terms in (3.1.7) do not depend on the decision variables, for the purposes of optimization we may redefine the profit origin and take as our value function

$$v(s_1,s_2;d_1,d_2) = -d_1^2 - d_2^2 + 2qd_1d_2 - d_1(a_1s_1 + a_2s_2)$$

$$- d_2(b_1s_1 + b_2s_2) . \tag{3.1.9}$$

Notice that if q were zero the value function in (3.1.9) would be additive, that is, of the form

$$v(\underset{\sim}{s};d_1,d_2) = v_1(\underset{\sim}{s};d_1) + v_2(\underset{\sim}{s};d_2) \ . \tag{3.1.10}$$

Thus, q is a measure of the interaction between the decision variables. The quantities within the parentheses in (3.1.9) are the deviations from mean values of the per unit output costs for outputs one and two. The parameters a_1, a_2, b_1, and b_2 measure the interaction in the value function between the various state and decision variables.

We shall assume that s_1 and s_2 have the bivariate normal distribution

$$\{s_1 s_2|\mathcal{E}\} = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{s_1}{\sigma_1}\right)^2 - 2\rho\frac{s_1 s_2}{\sigma_1\sigma_2} + \left(\frac{s_2}{\sigma_2}\right)^2\right]\right\} \ . \tag{3.1.11}$$

The quantity ρ is the correlation coefficient and indicates the extent to which the variables tend, on the average, to move together or in opposite directions. It is easy to show that

$$E(s_1 s_2) = \rho\sigma_1\sigma_2 \ , \tag{3.1.12}$$

$$E(s_2|s_1) = \rho\frac{\sigma_2}{\sigma_1} s_1 \ . \tag{3.1.13}$$

We shall now proceed to calculate the optimal decision rules and relative values under each of the possible perfect information with perfect flexibility structures. In each case the optimal decision rules were obtained by differentiation.

CASE 1: NO INFORMATION (NO FLEXIBILITY)

$$\langle v|\mathcal{E}\rangle = \max_{d_1 d_2} E[-d_1 - d_2 + 2qd_1d_2 - d_1(a_1s_1 + a_2s_2) - d_2(b_1s_1 + b_2s_2)]$$

$$= 0 \ , \tag{3.1.14}$$

$$d_1^* = d_2^* = 0 \ . \tag{3.1.15}$$

Because $\langle v|\mathcal{E}\rangle = 0$, the value of any information given flexibility structure, $\langle v|CF,\mathcal{E}\rangle - \langle v|\mathcal{E}\rangle$, will simply equal the expected payoff $\langle v|CF,\mathcal{E}\rangle$.

CASE 2: COMPLETE INFORMATION - COMPLETE FLEXIBILITY

$$\langle v|Cs_1s_2Fd_1d_2,\mathcal{E}\rangle = E\left[\max_{d_1d_2} v\right]$$

$$= \frac{(a_1^2+2qa_1b_1+b_1^2)\sigma_1^2 + 2[a_1a_2+q(a_1b_2+a_2b_1)+b_1b_2]\rho\sigma_1\sigma_2+(a_2^2+2qa_2b_2+b_2^2)\sigma_2^2}{4(1-q^2)} \tag{3.1.16}$$

$$d_1^* = -\frac{a_1s_1 + a_2s_2 + q(b_1s_1 + b_2s_2)}{2(1-q^2)} ,$$

$$d_2^* = -\frac{q(a_1s_1 + a_2s_2) + b_1s_1 + b_2s_2}{2(1-q^2)} . \tag{3.1.17}$$

Defining $c_1 = a_1s_1 + a_2s_2$, $c_2 = b_1s_1 + b_2s_2$ to be the deviations of the per unit output costs from their mean values,

$$d_1^* = -\frac{c_1 + qc_2}{2(1-q^2)} \quad \text{and} \quad d_2^* = -\frac{qc_1 + c_2}{2(1-q^2)} . \tag{3.1.18}$$

Observe that if the outputs are complements $(q > 0)$, [substitutes $(q < 0)$], the optimal output quantity falls as its cost rises and falls [rises] when the cost of the other output rises. Expressing the optimal decision rule as

$$d_1^* = -\frac{(a_1+qb_1)s_1+(a_2+qb_2)s_2}{2(1-q^2)} \quad \text{and} \quad d_2^* = -\frac{(qa_1+b_1)s_1+(qa_2+b_2)s_2}{2(1-q^2)} , \tag{3.1.19}$$

we see that the optimal quantities will normally fall when an input price is increased. In order for the reverse to be the case, we must have either little coupling between the output processes (a_2 or b_1 close to zero), or the products must be strong substitutes.

We may also write (3.1.16) as

$$\langle v | Cs_1 s_2 F d_1 d_2, \mathcal{E} \rangle = \frac{\sigma_{c_1}^2 + 2qr\sigma_{c_1}\sigma_{c_2} + \sigma_{c_2}^2}{4(1-q^2)} , \qquad (3.1.20)$$

where

$$\sigma_{c_1}^2 = E(c_1^2) ,$$

$$\sigma_{c_2}^2 = E(c_2^2) , \text{ and}$$

$$r = E(c_1 c_2)/\sigma_{c_1}\sigma_{c_2} .$$

Observe that the advantage of making decisions with full knowledge of costs is larger, the larger the product (qr) of the coefficients of interaction and correlation. If correlation between output costs is positive, the advantage is larger if the output products are complements $(q > 0)$. If correlation is negative, the advantage is larger if the outputs are substitutes $(q < 0)$.

CASE 3: PERFECT INFORMATION ON s_1 - COMPLETE FLEXIBILITY

$$\langle v | Cs_1 F d_1 d_2, \mathcal{E} \rangle = \underset{s_1}{E} \left[\max_{d_1 d_2} \underset{s_2}{E} (v | s_1) \right]$$

$$= \frac{(a_1^2+2qa_1b_1+b_1^2)\sigma_1^2 + 2[a_1b_1+q(a_1b_2+a_2b_1)+a_2b_2]\rho\sigma_1\sigma_2+(a_2^2+2qa_2b_2+b_2^2)\rho^2\sigma_2^2}{4(1-q^2)}$$

$$= \frac{\sigma_{c_1}^2 + 2qr\sigma_{c_1}\sigma_{c_2} + \sigma_{c_2}^2}{4(1-q^2)} - \frac{(1-\rho^2)\sigma_2^2}{4(1-q^2)} (a_2^2 + 2qa_2b_2 + b_2^2) , \qquad (3.1.21)$$

$$d_1^* = - \frac{a_1+a_2\rho \frac{\sigma_2}{\sigma_1} + q(b_1+b_2\rho \frac{\sigma_2}{\sigma_1})}{2(1-q^2)} s_1 \quad \text{and} \quad d_2^* = - \frac{q(a_1+a_2\rho \frac{\sigma_2}{\sigma_1})+b_1+b_2\rho \frac{\sigma_2}{\sigma_1}}{2(1-q^2)} s_1 .$$

(3.1.22)

The decision functions (3.1.17) are identical to (3.1.22) except that in the latter case the random variable s_2 is replaced by the estimate $E(s_2|s_1) = \rho(\sigma_2/\sigma_1)s_1$. This is an illustration of the well-known certainty equivalence principle: When the value function is quadratic, in the face of uncertainty the decision maker behaves as if he were certain that random variables take on their expected values.

Subtracting (3.1.21) from (3.1.16) we obtain the value of obtaining clairvoyance on s_1 and s_2 over the value of clairvoyance on s_1 alone.

$$\langle v|Cs_1s_2Fd_1d_2,\mathcal{E}\rangle - \langle v|Cs_1Fd_1d_2,\mathcal{E}\rangle = \frac{(1-\rho^2)\sigma_2^2}{4(1-q^2)}\,(a_2^2+2qa_2b_2+b_2^2) \ . \tag{3.1.23}$$

Clairvoyance on s_1 and s_2 over clairvoyance on s_1 is more valuable the smaller the correlation (positive or negative) and the larger the variance of s_2 . As a function of the interaction q , the value appears as illustrated in Fig. 3.2.

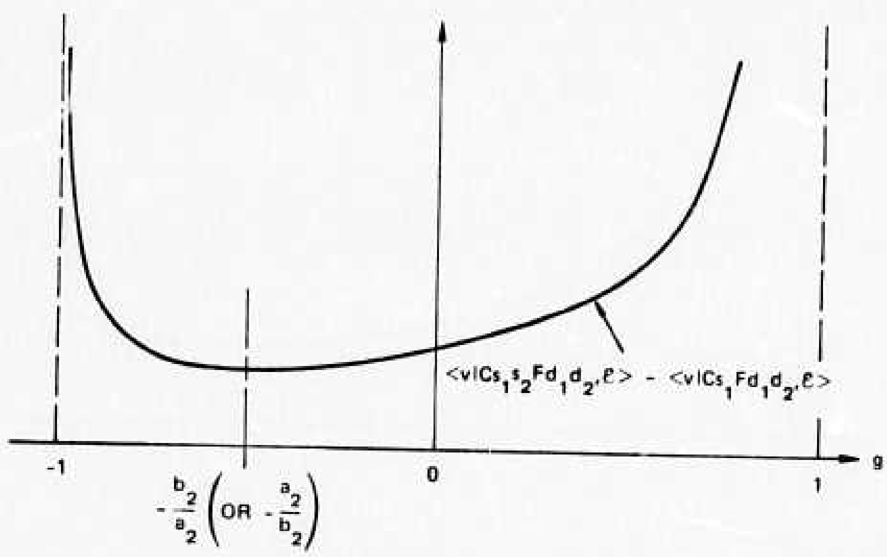


FIGURE 3.2 THE ADDITIONAL VALUE OF CLAIRVOYANCE ON s_1 AND s_2 OVER THE VALUE OF CLAIRVOYANCE ON s_1 ALONE AS A FUNCTION OF q IN THE FOUR-VARIABLE PRODUCTION PROBLEM

CASE 4: PERFECT INFORMATION ON s_2 - COMPLETE FLEXIBILITY

By symmetry,

$$\langle v|Cs_2Fd_1d_2,\mathcal{E}\rangle = \frac{\sigma_{c_1}^2+2qr\sigma_{c_1}\sigma_{c_2}+\sigma_{c_2}^2}{4(1-q^2)} - \frac{(1-\rho^2)\sigma_1^2}{4(1-q^2)}\,(a_1^2+2qa_1b_1+b_1^2)\ . \tag{3.1.24}$$

It is interesting to compare the value of clairvoyance on s_1 with the value of clairvoyance on s_2 . Subtracting (3.1.24) from (3.1.21),

$$\langle v|Cs_1Fd_1d_2,\mathcal{E}\rangle - \langle v|Cs_2Fd_1d_2,\mathcal{E}\rangle$$

$$= \frac{(1-\rho^2)}{4(1-q^2)}\,[(a_1^2+2qa_1b_1+b_1^2)\sigma_1^2-(a_2^2+2qa_2b_2+b_2^2)\sigma_2^2]\ . \tag{3.1.25}$$

Which information, s_1 or s_2 , is more valuable depends on the variances σ_1^2 and σ_2^2 and also on the relative proportions in which the inputs are used. If the input prices are strongly correlated, which of the two prices is ascertained is not important.

It is well known that the value of simultaneous information on several variables may be greater than or less than the sum of their individual values of information. Subtracting (3.1.24) and (3.1.21) from (3.1.16),

$$\langle v|Cs_1s_2Fd_1d_2,\mathcal{E}\rangle - \langle v|Cs_1Fd_1d_2,\mathcal{E}\rangle - \langle v|Cs_2Fd_1d_2,\mathcal{E}\rangle$$

$$= -\frac{(a_1^2+2qa_1b_1+b_1^2)\rho^2\sigma_1 + 2[a_1a_2+q(a_1b_2+a_2b_1)+b_1b_2]\rho\sigma_1\sigma_2}{4(1-q^2)}$$

$$+\frac{(a_2^2+2qa_2b_2+b_2^2)\rho^2\sigma_2^2}{4(1-q^2)}\ . \tag{3.1.26}$$

As a function of the correlation coefficient ρ , this appears as in Fig. 3.3. Observe that if the random variables are independent ($\rho = 0$), then the value of clairvoyance on s_1 , s_2 equals the sum of the individual values of clairvoyance on s_1 and on s_2 . The sign of

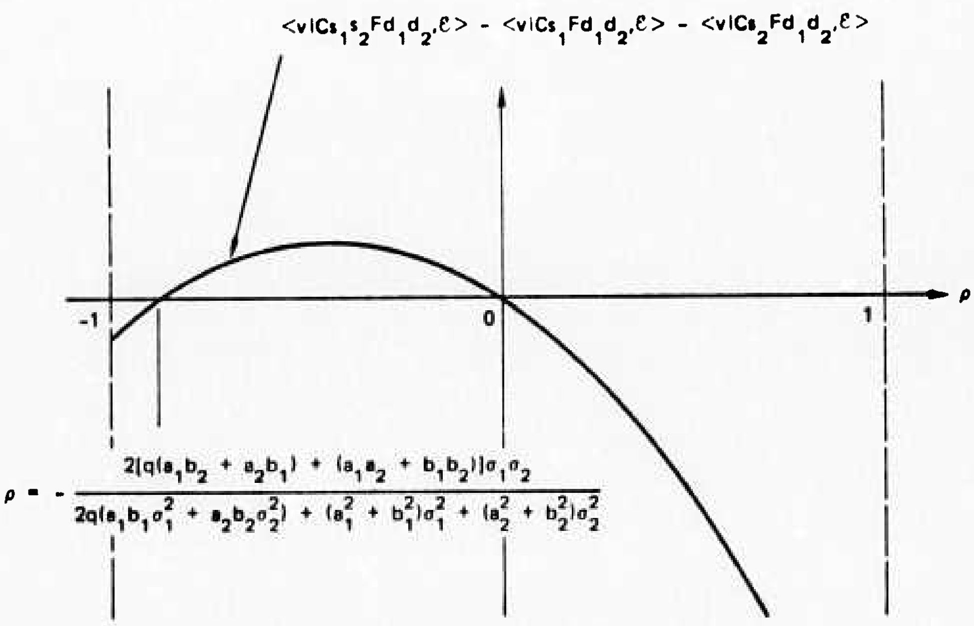


FIGURE 3.3 THE JOINT VALUE OF CLAIRVOYANCE MINUS THE INDIVIDUAL VALUES OF CLAIRVOYANCE IN THE FOUR-VARIABLE PRODUCTION PROBLEM

$\langle v|Cs_1s_2Fd_1d_2,\mathcal{E}\rangle - \langle v|Cs_1Fd_1d_2,\mathcal{E}\rangle - \langle v|Cs_2Fd_1d_2,\mathcal{E}\rangle$ depends on ρ and q as illustrated in Fig. 3.4. As we might expect, if correlation is high enough the sum of the values of individual information will exceed the value of joint information.

CASE 5: COMPLETE INFORMATION - PERFECT FLEXIBILITY ON d_1

$$\langle v|Cs_1s_2Fd_1,\mathcal{E}\rangle = \max_{d_2} E(\max_{d_1} v) = \frac{\sigma_{c_1}^2}{4}$$

$$= \frac{a_1^2\sigma_1^2+2a_1a_2\rho\sigma_1\sigma_2+a_2^2\sigma_2^2}{4}\,, \tag{3.1.27}$$

$$d_1^* = -\frac{a_1s_1+a_2s_2}{2} = -\frac{c_2}{2} \quad \text{and} \quad d_2^* = 0\,. \tag{3.1.28}$$

The value of flexibility on d_1, that is the value of choosing the quantity of output one after input prices become known, varies directly with the variance of the cost of output one. Retaining flexibility on the quantity of output one is a means of compensating for cost changes in that output, the optimum rule being to decrease output by one-half of the net increase in output costs. The value of that compensation ability varies directly with uncertainty in the quantity to be compensated for. The more uncertain a variable for which we may compensate, the more valuable is the compensation. Notice that the result is independent of the degree of interaction q between the decision variables.

CASE 6: COMPLETE INFORMATION - PERFECT FLEXIBILITY ON d_2

By symmetry

$$\langle v|Cs_1s_2Fd_1,\mathcal{E}\rangle = \frac{\sigma_{c_2}^2}{4} = \frac{b_1^2\sigma_1^2+2b_1b_2\rho\sigma_1\sigma_2+b_2^2\sigma_2^2}{4}\,. \tag{3.2.29}$$

Comparing (3.1.27) and (3.1.29) we see that it is more valuable to retain flexibility on the output whose costs are more variable; that is, other things being equal, compensation should be reserved for variables whose values are most uncertain. Subtracting (3.1.27) and (3.1.29)

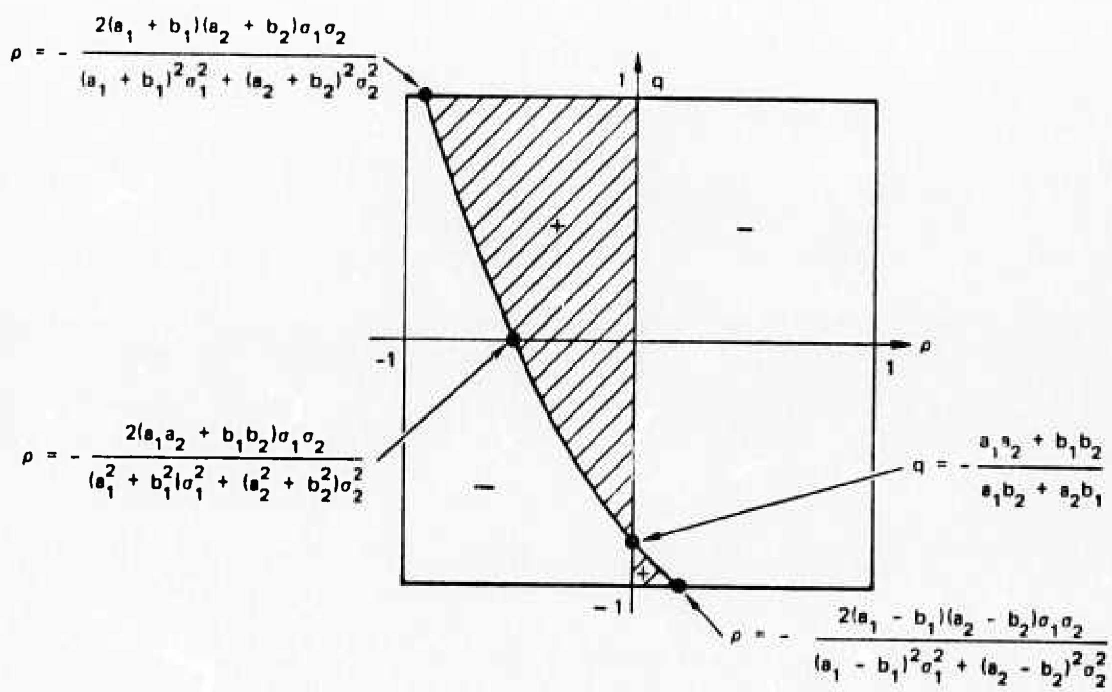


FIGURE 3.4 THE SIGN OF $\langle v|Cs_1s_2Fd_1d_2,\mathcal{E}\rangle - \langle v|Cs_1Fd_1d_2,\mathcal{E}\rangle - \langle v|Cs_2Fd_1d_2,\mathcal{E}\rangle$ AS A FUNCTION OF ρ AND q FOR THE FOUR-VARIABLE PRODUCTION PROBLEM

from (3.1.16),

$$\langle v|Cs_1s_2Fd_1d_2,\mathcal{E}\rangle - \langle v|Cs_1s_2Fd_1,\mathcal{E}\rangle - \langle v|Cs_1s_2Fd_2,\mathcal{E}\rangle$$

$$= \frac{q^2\sigma_{c_1}^2 + 2qr\sigma_{c_1}\sigma_{c_2} + q^2\sigma_{c_2}^2}{4(1-q^2)}$$

$$= \frac{(q^2a_1^2+2qa_1b_1+q^2b_1^2)\sigma_1^2 + 2[q^2a_1a_2+q(a_1b_2+a_2b_1)+q^2b_1b_2]\rho\sigma_1\sigma_2}{4(1-q^2)}$$

$$+ \frac{(q^2a_2^2 + 2qa_2b_2+q^2b_2^2)\sigma_2^2}{4(1-q^2)} \quad . \qquad (3.1.30)$$

As a function of the interaction coefficient q this appears as in Fig. 3.5. Observe that if there is no interaction between the decision variables $(q = 0)$, then the value of flexibility on d_1 and d_2 equals the sum of the individual values of flexibility on d_1 and on d_2. If the input cost random variables are such that there is no correlation between output costs $(r = 0)$, then the value of flexibility on both d_1 and d_2 is at least as valuable as the sum of the values of flexibility on each individual decision variable. In general, however, the value of flexibility shares that same (perhaps somewhat perplexing) characteristic of the value of information; $\langle v|Cs_1s_2Fd_1d_2,\mathcal{E}\rangle$ may be greater than or less than the sum of $\langle v|Cs_1s_2Fd_1,\mathcal{E}\rangle$ and $\langle v|Cs_1s_2Fd_2,\mathcal{E}\rangle$. The sign of $\langle v|Cs_1s_2Fd_1d_2,\mathcal{E}\rangle - \langle v|Cs_1s_2Fd_1,\mathcal{E}\rangle$ depends on ρ and q as illustrated in Fig. 3.6. If decision variable interaction is high enough, we can expect the value of joint flexibility to exceed the sum of the values of individual flexibility.

CASE 7: PERFECT INFORMATION ON s_1 - PERFECT FLEXIBILITY ON d_1

$$\langle v|Cs_1Fd_1,\mathcal{E}\rangle = \max_{d_2} E[\max_{s_1 \; d_1} E(v|s_1)]_{s_2}$$

$$= \frac{\sigma_{c_1}^2}{4} - (1-\rho^2)\frac{a_2^2}{4}\sigma_2^2 = \frac{a_1^2\sigma_1^2 + 2a_1a_2\rho\sigma_1\sigma_2 + a_2^2\rho^2\sigma_2^2}{4} \quad , \qquad (3.1.31)$$

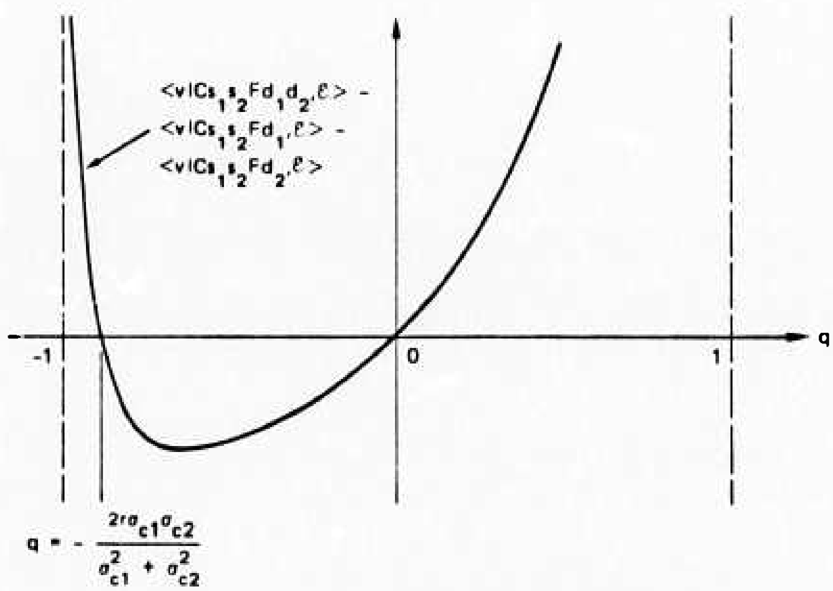


FIGURE 3.5 THE JOINT VALUE OF FLEXIBILITY MINUS THE INDIVIDUAL VALUES OF FLEXIBILITY IN THE FOUR-VARIABLE PRODUCTION PROBLEM

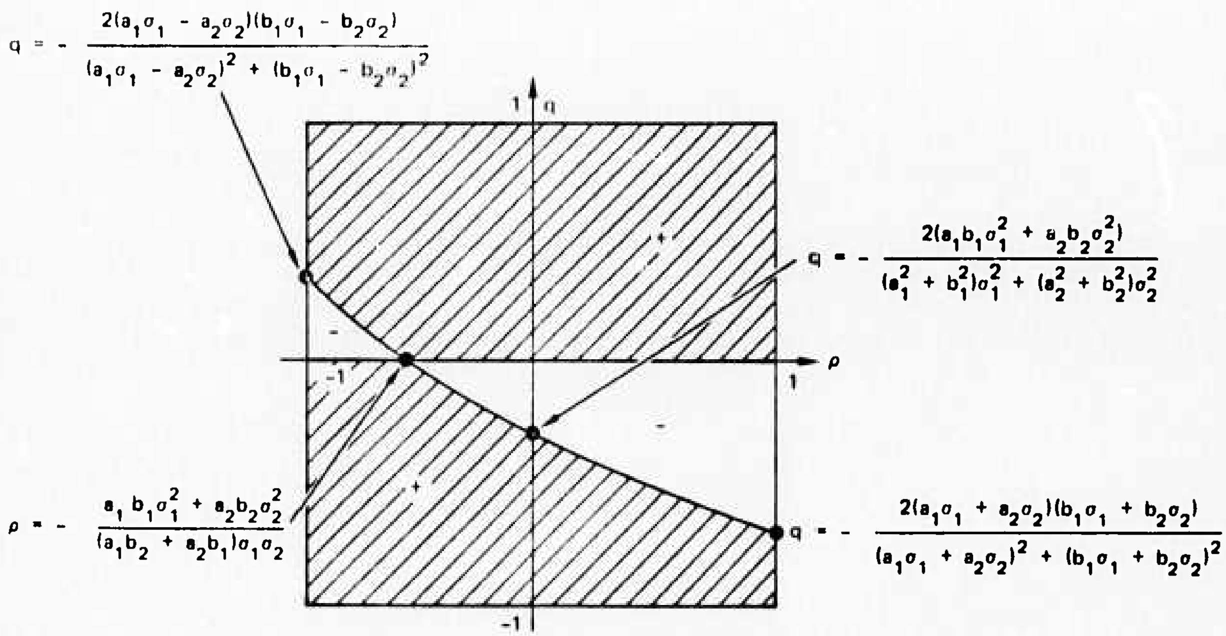


FIGURE 3.6 THE SIGN OF $\langle v|Cs_1s_2Fd_1d_2,\mathcal{E}\rangle - \langle v|Cs_1s_2Fd_1,\mathcal{E}\rangle - \langle v|Cs_1s_2Fd_2,\mathcal{E}\rangle$ AS A FUNCTION OF ρ AND q FOR THE FOUR-VARIABLE PRODUCTION PROBLEM

$$d_1^* = -\frac{a_1 + a_2 \rho \frac{\sigma_2}{\sigma_1}}{2} s_1 \quad \text{and} \quad d_2^* = 0 \;. \tag{3.1.32}$$

Subtracting (3.1.31) from (3.1.27),

$$\langle v|Cs_1 s_2 Fd_1, \mathcal{E}\rangle - \langle v|Cs_1 Fd_1, \mathcal{E}\rangle = (1-\rho^2)\,\frac{a_2^2}{4}\,\sigma_2^2 \;. \tag{3.1.33}$$

The value of clairvoyance on s_1 and s_2 over clairvoyance on s_1 when that information may be used only in setting d_1 is independent of the interaction q and the parameters describing process number two. It is smaller the stronger the correlation ρ.

CASE 8: PERFECT INFORMATION ON s_1 - PERFECT FLEXIBILITY ON d_2

By symmetry,

$$\langle v|Cs_1 Fd_2, \mathcal{E}\rangle = \frac{\sigma_{c_2}^2}{4} - (1-\rho^2)\,\frac{b_2^2}{4}\,\sigma_2^2 = \frac{b_1^2\sigma_1^2 + 2b_1 b_2 \rho\sigma_1\sigma_2 + b_2^2\rho^2\sigma_2^2}{4} \;. \tag{3.1.34}$$

3.2 THE N x M QUADRATIC DECISION PROBLEM

In this section we shall generalize the results on the four-variable quadratic problem to the case of a quadratic decision problem with any finite number of state and decision variables. We take for our general framework the basic decision model of Fig. 1.4 with the following additional assumptions:

1. The decision variables d_j are unconstrained, $-\infty < d_j < \infty$.
2. The decision maker's value function $v(\underline{s},\underline{d})$ is a quadratic function in the s_i and d_j such that for every $\underline{s}$, $v(\underline{s},\underline{d})$ has a unique maximum with respect to $\underline{d}$.
3. The decision maker's utility function is a linear function of value.

We may think of the quadratic value function as an approximation to an arbitrary smooth value function $\tilde{v}(\underline{s},\underline{d})$ as follows. Expand $\tilde{v}(\underline{s},\underline{d})$ in a Taylor series about the point $\underline{o} = (\underline{s}^o,\underline{d}^o)$ neglecting higher order terms:

$$\tilde{v}(\underline{s},\underline{d}) \cong v(\underline{s},\underline{d})$$

$$= \tilde{v}(\underline{s}^o,\underline{d}^o) + \sum_i \left.\frac{\partial \tilde{v}}{\partial s_i}\right|_{\underline{o}} (s_i - s_i^o) + \frac{1}{2}\sum_{ij} \left.\frac{\partial^2 \tilde{v}}{\partial s_i \partial s_j}\right|_{\underline{o}} (s_i - s_i^o)(s_j - s_j^o)$$

$$+ \sum_{ij} \left.\frac{\partial^2 \tilde{v}}{\partial s_i \partial d_j}\right|_{\underline{o}} (s_i - s_i^o)(d_j - d_j^o) + \sum_j \left.\frac{\partial \tilde{v}}{\partial d_j}\right|_{\underline{o}} (d_j - d_j^o)$$

$$+ \frac{1}{2}\sum_{ij} \left.\frac{\partial^2 \tilde{v}}{\partial d_i \partial d_j}\right|_{\underline{o}} (d_i - d_i^o)(d_j - d_j^o) \ . \qquad (3.2.1)$$

By suitably defining the real number a, vectors $\underline{b}$ and $\underline{r}$, and matrices W, T, and Q, this may be written

$$v(\underline{s},\underline{d}) = a + \underline{b}'\underline{s} + \frac{1}{2}\underline{s}'W\underline{s} + \underline{s}'T\underline{d} + \underline{r}'\underline{d} + \frac{1}{2}\underline{d}'Q\underline{d} \ , \qquad (3.2.2)$$

where

$$T = \left[\left.\frac{\partial^2 \tilde{v}}{\partial s_i \partial d_j}\right|_{\underline{o}}\right]_{i=1,\ldots,n,\ j=1,\ldots,m} \qquad (3.2.3)$$

is a matrix whose elements measure the interaction between various state and decision variables, and

$$Q = \left[\left.\frac{\partial^2 \tilde{v}}{\partial d_i \partial d_j}\right|_{\underline{o}}\right]_{i=1,\ldots,n,\ j=1,\ldots,m} \qquad (3.2.4)$$

is a symmetric matrix measuring the interaction between decision variables. Assumption (2) above implies that Q is negative definite.

AN EXPRESSION FOR THE EVPIGPF

We are interested in comparing alternative information with flexibility structures in terms of the maximum expected payoff that can be derived from their use. To characterize the various structures we use the following notation. Let $N = \{1,\ldots,n\}$ and $M = \{1,\ldots,m\}$ be the respective sets of state and decision variable indices. Define $I \subset N$ to be the set of indices of those state variables upon which information is to be obtained, and let $J \subset M$ denote the indices of decision variables for which flexibility is available. $\bar{I}$ and $\bar{J}$ will denote the complements within N and M of the sets I and J, respectively. $Cs_I Fd_J$ will denote the information structure within which the decision maker has clairvoyance on state variables s_i, $i \in I$ and flexibility on decision variables d_j, $j \in J$.

It will be convenient to take as the origin for measurement of expected payoff the maximum expected payoff for the null system which provides no information nor flexibility. Because of assumption (3) above, the value of an information with flexibility structure $Cs_I Fd_J$ will be given by

$$\langle v_{Cs_I Fd_J} | \mathcal{E} \rangle = \langle v | Cs_I Fd_J, \mathcal{E} \rangle - \langle v | \mathcal{E} \rangle . \tag{3.2.5}$$

Since we will be interested only in the relative values of using various information structures, we ignore the first three terms in (3.2.2). We shall also find it convenient to measure the decision variable setting as a deviation from the best deterministic decision rule; that is, from the decision setting that would be optimal if state variables were to take on their expected values. Replacing $\underline{s}$ by $E(\underline{s})$ and setting the gradient of (3.2.2) to zero, we get

$$\hat{\underline{d}} = -Q^{-1}[T'E(\underline{s}) + \underline{r}] . \tag{3.2.6}$$

Defining $\delta\underline{d}$ as the deviation from the deterministic optimum, $\underline{d} = \hat{\underline{d}} + \delta\underline{d}$, the value function (3.2.2) may be written

$$v = [s'-E(\underline{s}')]\, T\, \delta\underline{d} + \frac{1}{2}\, \delta\underline{d}'Q\, \delta\underline{d} + \text{terms independent of } \delta\underline{d} . \tag{3.2.7}$$

Thus, there is no loss in generality if we take

$$v(\underline{s},\underline{d}) = \underline{s}'T\, \underline{d} + \frac{1}{2}\, \underline{d}'Q\, \underline{d} \tag{3.2.8}$$

with $E(\underline{s}) = \underline{0}$, if state variables are assumed to be measured from their mean values and decision variables are assumed measured as deviations from the values that would be optimal if state variables were at their means.

In order to state the major theorem of this section we need to define some supplemental notation. For a given structure $Cs_I Fd_J$, let T_{IJ} denote the matrix $[t_{ij}]_{i\in I, j\in J}$ of those elements t_{ij} of T such that i is in I and j is in J , and similarly define T_{NJ} , Q_{JJ} , $Q_{\overline{J}\overline{J}}$, etc. T'_{NJ} and Q^{-1}_{JJ} will be taken to mean the transpose of T_{NJ} and the inverse of Q_{JJ} respectively. Also, let $\underline{s}_I$ denote the vector of those components s_i of $\underline{s}$ such that $i \in I$, and similarly define $\underline{d}_J$ and $\underline{d}_{\overline{J}}$. Then, subject to the various assumptions made above, we have the

<u>THEOREM</u>: For any information-flexibility structure $Cs_I Fd_J$, the optimal decision strategy $\underline{d}^*$ is given by

$$\underline{d}^*_{\overline{J}} = -(Q_{\overline{J}\overline{J}} - Q_{\overline{J}J}Q^{-1}_{JJ}Q_{J\overline{J}})^{-1}(T'_{\overline{I}\overline{J}} - Q_{\overline{J}J}Q^{-1}_{JJ}T'_{\overline{I}J})E(\underline{x}_{\overline{I}}) , \tag{3.2.9}$$

$$\begin{aligned}\underline{d}^*_J = &-(Q_{JJ} - Q_{J\overline{J}}Q^{-1}_{\overline{J}\overline{J}}Q_{\overline{J}J})^{-1}(T'_{\overline{I}J} - Q_{J\overline{J}}Q^{-1}_{\overline{J}\overline{J}}T'_{\overline{I}\overline{J}})E(\underline{x}_{\overline{I}}) \\ &-Q^{-1}_{JJ}[T'_{IJ}\underline{s}_I + T'_{\overline{I}J}[\underline{x}_{\overline{I}} - E(\underline{x}_{\overline{I}})]] ,\end{aligned} \tag{3.2.10}$$

and the corresponding expected value of the structure is

$$\begin{aligned}\langle V_{Cs_I Fd_J}|\mathcal{E}\rangle = -\frac{1}{2}\,\mathrm{trace}\Big\{&T_{NJ}Q^{-1}_{JJ}T'_{NJ}E(\underline{xx}') \\ &+(T_{\overline{I}\overline{J}} - T_{\overline{I}J}Q^{-1}_{JJ}Q_{J\overline{J}})(Q_{\overline{J}\overline{J}} - Q_{\overline{J}J}Q^{-1}_{JJ}Q_{J\overline{J}})^{-1}(T'_{\overline{I}\overline{J}} - Q_{\overline{J}J}Q^{-1}_{JJ}T'_{\overline{I}J})E(\underline{x}_{\overline{I}})E(\underline{x}_{\overline{I}})\Big\} ,\end{aligned} \tag{3.2.11}$$

where $\underline{x} = E(\underline{s}|\underline{s}_I)$.

<u>PROOF</u>: We first note that the expected payoff using the null structure is

$$\langle v|\mathcal{E}\rangle = \max_{\underline{d}} E(v) = \max_{\underline{d}} (\frac{1}{2}\,\underline{d}'Q\,\underline{d}) = 0 , \tag{3.2.12}$$

since Q is negative definite. Therefore, the value of the structure $Cs_I Fd_J$ will be $\langle v|Cs_I Fd_J,\mathcal{E}\rangle$. Now,

$$\langle v|Cs_I Fd_J,\mathcal{E}\rangle = \max_{\underline{d}_{\bar{J}}} E[\max_{\underline{d}_J} E(v|\underline{s}_I)]$$

$$= \max_{\underline{d}_{\bar{J}}} E\left\{\max_{\underline{d}_J}[\underline{x}'T_{NJ}\underline{d}_J+\underline{x}'T_{N\bar{J}}\underline{d}_{\bar{J}}+\frac{1}{2}\underline{d}_J'Q_{JJ}\underline{d}_J+\underline{d}_J'Q_{J\bar{J}}\underline{d}_{\bar{J}} + \frac{1}{2}\underline{d}_{\bar{J}}'Q_{\bar{J}\bar{J}}\underline{d}_{\bar{J}}]\right\} , \tag{3.2.13}$$

where $\underline{x} = E(\underline{s}|\underline{s}_I)$ denotes the vector of conditional means. Since Q is negative definite, the submatrix Q_{JJ} is negative definite also. Taking the gradient with respect to d_J of the quantity within the inner bracket and setting the result equal to zero we obtain

$$\underline{d}_J^* = -Q_{JJ}^{-1}[T_{NJ}'\underline{x}+Q_{J\bar{J}}\underline{d}_{\bar{J}}] . \tag{3.2.14}$$

Substituting (3.2.14) into (3.2.13) gives

$$\langle v|Cs_I Fd_J,\mathcal{E}\rangle = \max_{\underline{d}_{\bar{J}}}\left\{-\frac{1}{2}E[\underline{x}'T_{NJ}Q_{JJ}^{-1}T_{NJ}'\underline{x}]-\underline{d}_{\bar{J}}'(Q_{\bar{J}J}Q_{JJ}^{-1}T_{NJ}'-T_{N\bar{J}}')E(\underline{x})\right.$$
$$\left.+\frac{1}{2}\underline{d}_{\bar{J}}'(Q_{\bar{J}\bar{J}}-Q_{\bar{J}J}Q_{JJ}^{-1}Q_{J\bar{J}})\underline{d}_{\bar{J}}\right\} . \tag{3.2.15}$$

Let us denote the inverse of Q by R and partition R in conformity with the partitioning of Q . Then we have

$$\begin{bmatrix} Q_{JJ} & Q_{J\bar{J}} \\ Q_{\bar{J}J} & Q_{\bar{J}\bar{J}} \end{bmatrix}\begin{bmatrix} R_{JJ} & R_{J\bar{J}} \\ R_{\bar{J}J} & R_{\bar{J}\bar{J}} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} , \tag{3.2.16}$$

which leads to the equations

$$Q_{JJ}R_{JJ} + Q_{J\bar{J}}R_{\bar{J}J} = I ,$$

$$Q_{JJ}R_{J\bar{J}} + Q_{J\bar{J}}R_{\bar{J}\bar{J}} = 0 ,$$

$$Q_{\bar{J}J}R_{JJ} + Q_{\bar{J}\bar{J}}R_{\bar{J}J} = 0 ,$$

$$Q_{\bar{J}J}R_{J\bar{J}} + Q_{\bar{J}\bar{J}}R_{\bar{J}\bar{J}} = I , \tag{3.2.17}$$

in which I and 0 denote identity and zero matrices of appropriate dimension. Solving these equations for the elements of R leads to the equations

$$
\begin{aligned}
R_{JJ} &= (Q_{JJ}-Q_{J\bar{J}}Q_{\bar{J}\bar{J}}^{-1}Q_{\bar{J}J})^{-1} , \\
R_{J\bar{J}} &= -Q_{JJ}^{-1}Q_{J\bar{J}}R_{\bar{J}\bar{J}} = -R_{JJ}Q_{J\bar{J}}Q_{\bar{J}\bar{J}}^{-1} , \\
R_{\bar{J}J} &= -Q_{\bar{J}\bar{J}}^{-1}Q_{\bar{J}J}R_{JJ} = -R_{\bar{J}\bar{J}}Q_{\bar{J}J}Q_{JJ}^{-1} , \\
R_{\bar{J}\bar{J}} &= (Q_{\bar{J}\bar{J}}-Q_{\bar{J}J}Q_{JJ}^{-1}Q_{J\bar{J}})^{-1} .
\end{aligned}
\tag{3.2.18}
$$

These relations are symmetric in that we can exchange the symbols R and **Q** . For example, $Q_{JJ} = (R_{JJ}-R_{J\bar{J}}R_{\bar{J}\bar{J}}^{-1}R_{\bar{J}J})^{-1}$

Using (3.2.18), (3.2.15) may be written

$$
\begin{aligned}
\langle v|Cs_I Fd_J,\mathcal{E}\rangle &= -\frac{1}{2}E(\underline{x}'T_{NJ}Q_{JJ}^{-1}T_{NJ}'\underline{x}) \\
&\quad - \max_{\underline{d}_{\bar{J}}}\Big[\underline{d}_{\bar{J}}R_{\bar{J}\bar{J}}^{-1}(R_{\bar{J}J}T_{NJ}'-R_{\bar{J}\bar{J}}T_{N\bar{J}}')E(\underline{x})+\frac{1}{2}\underline{d}_{\bar{J}}'R_{\bar{J}\bar{J}}^{-1}\underline{d}_{\bar{J}}\Big] .
\end{aligned}
\tag{3.2.19}
$$

Since **Q** is negative definite, so is R , and therefore, so are $R_{\bar{J}\bar{J}}$ and $R_{\bar{J}\bar{J}}^{-1}$. Hence, the maximization in (3.2.19) may be achieved by differentiation. We obtain

$$
\begin{aligned}
\underline{d}_{\bar{J}}^{*} &= -(R_{\bar{J}J}T_{NJ}' + R_{\bar{J}\bar{J}}T_{N\bar{J}}')E(\underline{x}) \\
&= -R_{\bar{J}M}T'E(\underline{x}) .
\end{aligned}
\tag{3.2.20}
$$

Substituting this expression into (3.2.14) and (3.2.19) yields

$$
\begin{aligned}
\underline{d}_{J}^{*} &= -Q_{JJ}^{-1}T_{NJ}'\underline{x} + Q_{JJ}^{-1}[Q_{J\bar{J}}R_{\bar{J}J}T_{NJ}' + Q_{J\bar{J}}R_{\bar{J}\bar{J}}T_{N\bar{J}}']E(\underline{x}) \\
&= -Q_{JJ}^{-1}T_{NJ}'\underline{x} + Q_{JJ}^{-1}[(I-Q_{JJ}R_{JJ})T_{NJ}' - Q_{JJ}R_{J\bar{J}}T_{N\bar{J}}']E(\underline{x}) \\
&= -Q_{JJ}^{-1}T_{NJ}'[\underline{x} - E(\underline{x})] - R_{JM}T'E(\underline{x})
\end{aligned}
\tag{3.2.21}
$$

and

$$\langle v \,|\, Cs_I Fd_J, \mathcal{E}\rangle = -\frac{1}{2}\, E(\underline{x}' T_{NJ} Q_{JJ}^{-1} T_{NJ}' \underline{x}) - \frac{1}{2} E(\underline{x}') T R_{M\bar{J}} R_{\bar{J}\bar{J}}^{-1} R_{\bar{J}M} T' E(\underline{x}) \;, \tag{3.2.22}$$

or, since $\underline{x}_I = E(\underline{s}_I | \underline{s}_I) = \underline{s}_I$, $E(\underline{x}_I) = \underline{0}$

$$\underline{d}_{\bar{J}}^* = -R_{\bar{J}M} T_{\bar{I}M}' E(\underline{x}_{\bar{I}}) \;, \tag{3.2.23}$$

$$\underline{d}_J^* = -Q_{JJ}^{-1} T_{IJ}' \underline{s}_I - Q_{JJ}^{-1} T_{\bar{I}J}' [\underline{x}_{\bar{I}} - E(\underline{x}_{\bar{I}})] - R_{JM} T_{\bar{I}M}' E(\underline{x}_{\bar{I}}) \;, \tag{3.2.24}$$

$$\begin{aligned} \langle v \,|\, Cs_I Fd_J, \mathcal{E}\rangle &= -\frac{1}{2} E(\underline{x}' T_{NJ} Q_{JJ}^{-1} T_{NJ}' \underline{x}) - \frac{1}{2} E(\underline{x}_{\bar{I}}') T_{\bar{I}M} R_{M\bar{J}} R_{\bar{J}\bar{J}}^{-1} R_{\bar{J}M} T_{\bar{I}M}' E(\underline{x}_{\bar{I}}) \\ &= -\frac{1}{2} \operatorname{trace}\left\{ T_{NJ} Q_{JJ}^{-1} T_{NJ}' E(\underline{xx}') + T_{\bar{I}M} R_{M\bar{J}} R_{\bar{J}\bar{J}}^{-1} R_{\bar{J}M} T_{\bar{I}M}' E(\underline{x}_{\bar{I}}) E(\underline{x}_{\bar{I}}') \right\} . \end{aligned} \tag{3.2.25}$$

Finally, (3.2.23), (3.2.24), and (3.2.25) may be put in the form of the theorem using the identities

$$\begin{aligned} R_{JM} T_{\bar{I}M}' &= R_{JJ} T_{\bar{I}J}' + R_{J\bar{J}} T_{\bar{I}\bar{J}}' \\ &= R_{JJ}(T_{\bar{I}J}' + R_{JJ}^{-1} R_{J\bar{J}} T_{\bar{I}\bar{J}}') = R_{JJ}(T_{\bar{I}J}' - Q_{J\bar{J}} Q_{\bar{J}\bar{J}}^{-1} T_{\bar{I}\bar{J}}') \;, \end{aligned} \tag{3.2.26}$$

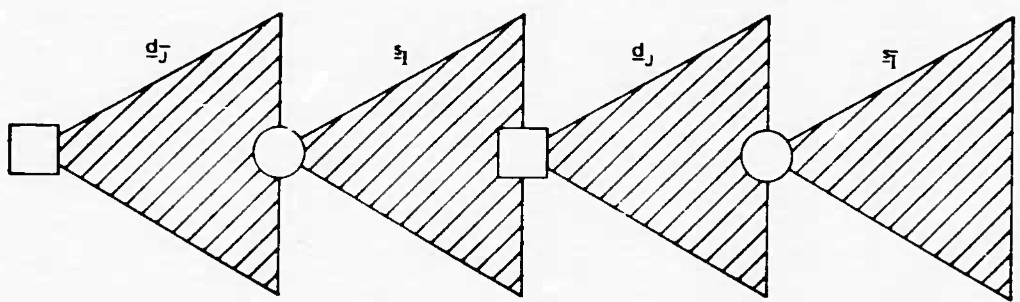
$$\begin{aligned} R_{\bar{J}M} T_{\bar{I}M}' &= R_{\bar{J}\bar{J}} T_{\bar{I}\bar{J}}' + R_{\bar{J}J} T_{\bar{I}J}' \\ &= R_{\bar{J}\bar{J}}(T_{\bar{I}\bar{J}}' + R_{\bar{J}\bar{J}}^{-1} R_{\bar{J}J} T_{\bar{I}J}') = R_{\bar{J}\bar{J}}(T_{\bar{I}\bar{J}}' - Q_{\bar{J}J} Q_{JJ}^{-1} T_{\bar{I}J}') \;, \end{aligned} \tag{3.2.27}$$

$$R_{JJ} = (Q_{JJ} - Q_{J\bar{J}} Q_{\bar{J}\bar{J}}^{-1} Q_{\bar{J}J})^{-1} \;, \tag{3.2.28}$$

$$R_{\bar{J}\bar{J}} = (Q_{\bar{J}\bar{J}} - Q_{\bar{J}J} Q_{JJ}^{-1} Q_{J\bar{J}})^{-1} \;. \tag{3.2.29}$$

QED.

The decision problem that the theorem solves is illustrated in Fig. 3.7. It is interesting to observe that a form of the certainty equivalence principle continues to apply. If $\underline{s}$ were known exactly,



OUTCOME VALUE

$$v = \underline{s}_I' T_{IJ} \underline{d}_J + \underline{s}_{\bar{I}}' T_{\bar{I}J} \underline{d}_J + \underline{s}_I' T_{I\bar{J}} \underline{d}_{\bar{J}} + \underline{s}_{\bar{I}}' T_{\bar{I}\bar{J}} \underline{d}_{\bar{J}}$$
$$+ \tfrac{1}{2}(\underline{d}_J' Q_{JJ} \underline{d}_J + \underline{d}_{\bar{J}}' Q_{\bar{J}J} \underline{d}_J + \underline{d}_J' Q_{J\bar{J}} \underline{d}_{\bar{J}} + \underline{d}_{\bar{J}}' Q_{\bar{J}\bar{J}} \underline{d}_{\bar{J}})$$

FIGURE 3.7 THE QUADRATIC DECISION PROBLEM WITH CLAIRVOYANCE ON s_I AND PERFECT FLEXIBILITY ON d_J

the optimum setting for $\underline{d}_{\bar{J}}$ would be

$$\underline{d}^*_{\bar{J}} = -R_{\bar{J}M}T'\underline{s} \ . \tag{3.2.30}$$

If we replace $\underline{s}$ by its prior expectation $[\underline{0},E(\underline{x}_{\bar{I}})]$, we get

$$\underline{d}^*_{\bar{J}} = -R_{JM}T'_{\bar{I}M}E(\underline{x}_{\bar{I}}) \ , \tag{3.2.31}$$

which by (3.2.23) is the optimal setting under uncertainty. Given this setting for $\underline{d}_{\bar{J}}$, the optimal setting for $\underline{d}_J$ given $\underline{s}$ would be

$$\underline{d}^*_J = -Q^{-1}_{JJ}T'_{NJ}\underline{s} + Q^{-1}_{JJ}T'_{\bar{I}J}E(\underline{x}_{\bar{I}}) - R_{JM}T'_{\bar{I}M}E(\underline{x}_{\bar{I}}) \ . \tag{3.2.32}$$

Replacing $\underline{s}$ with its posterior expectation $[\underline{s}_I,\underline{x}_{\bar{I}}]$, we obtain Eq. (3.2.24),

$$\underline{d}^*_J = -Q^{-1}_{JJ}T'_{IJ}\underline{s}_I - Q^{-1}_{JJ}T'_{\bar{I}J}[\underline{x}_{\bar{I}}-E(\underline{x}_{\bar{I}})] - R_{JM}T'_{\bar{I}M}E(\underline{x}_{\bar{I}}) \ , \tag{3.2.33}$$

the optimal setting for $\underline{d}_J$.

<u>COROLLARY 1</u>: Under the structure Cs_IFd_J

$$\underline{d}^*_{\bar{J}} = \underline{0} \ , \tag{3.2.34}$$

$$\underline{d}^*_J = -Q^{-1}_{JJ}T'_{NJ}\underline{x} \ , \tag{3.2.35}$$

$$\langle V_{Cs_IFd_J}|\mathcal{E}\rangle = -\frac{1}{2}\text{trace}\left\{T_{NJ}Q^{-1}_{JJ}T'_{NJ}E(\underline{xx}')\right\} \tag{3.2.36}$$

if any of the following conditions hold:

(a) $E(\underline{x}_{\bar{I}}) = \underline{0}$

(b) $\bar{J} = \emptyset$ (Complete flexibility)

(c) $\bar{I} = \emptyset$ (Complete information) .

<u>PROOF</u>: For condition (a) the proof follows trivially from (3.2.23), (3.2.24), and (3.2.25). That the reduction results for conditions (b) and (c) is easily deduced by reviewing the proof for the above theorem under each condition. **QED.**

COROLLARY 2: Under the structure $Cs_I Fd_J$

$$\underline{d}_J = \underline{0} \ , \tag{3.2.37}$$

$$\underline{d}_J^* = -Q_{JJ}^{-1} T_{IJ}' \underline{s}_I \ , \tag{3.2.38}$$

$$\langle v_{Cs_I Fd_J} | \mathcal{E} \rangle = -\frac{1}{2} \text{trace} \left\{ T_{IJ} Q_{JJ}^{-1} T_{IJ}' E(\underline{s}_I \underline{s}_I') \right\} \ , \tag{3.2.39}$$

if any of the following conditions hold:

(a) $\underline{x}_I = 0$

(b) $T_{IM} = [0]$

(c) $E(\underline{x}_I) = \underline{0}$ and $T_{IJ} = [0]$.

PROOF: The proof follows by direct substitution into (3.2.23), (3.2.24), and (3.2.25). QED.

ADDITIVITY CHARACTERISTICS OF THE EVPIGPF

For the following two corollaries we shall assume in addition that the conditional expectation of $\underline{s}$ is a linear function of the observable state variables. Condition (a) of Corollary 1 holds in this case; we may generalize two of the results obtained in the four-variable quadratic example of Section 3.1.

COROLLARY 3: Suppose the random variables composing the vector $\underline{s}_I$ upon which clairvoyance is available may be partitioned into two vectors $\underline{s}_{I1}$ and $\underline{s}_{I2}$ that are independent. Then

$$\langle v_{Cs_I Fd_J} | \mathcal{E} \rangle = \langle v_{Cs_{I1} Fd_J} | \mathcal{E} \rangle + \langle v_{Cs_{I2} Fd_J} | \mathcal{E} \rangle \ . \tag{3.2.40}$$

PROOF: By assumption, $\underline{x} = E(\underline{s} | \underline{s}_I) = D\underline{s}_I$ for some matrix D . Denoting the covariance matrix of $\underline{s}_I$ by C_{II} , (3.2.36) becomes

$$\langle v_{Cs_I Fd_J} | \mathcal{E} \rangle = -\frac{1}{2} \text{trace} \left\{ D' T_{NJ} Q_{JJ}^{-1} T_{NJ} D C_{II} \right\} \ . \tag{3.2.41}$$

For convenience in what follows we shall assume that the variables have been ordered so that

$$\underline{s} = \begin{bmatrix} \underline{s}_{I1} \\ \underline{s}_{I2} \\ \underline{s}_{\bar{I}} \end{bmatrix} . \qquad (3.2.42)$$

The independence of $\underline{s}_{I1}$ and $\underline{s}_{I2}$ imply

$$C_{II} = \begin{bmatrix} C_{I1I1} & 0 \\ 0 & C_{I2I2} \end{bmatrix} , \qquad (3.2.43)$$

$$D = \begin{bmatrix} I & 0 \\ 0 & I \\ D_1 & D_2 \end{bmatrix} , \qquad (3.2.44)$$

where I and 0 are identity and zero matrices. Similarly, with $E(\underline{s}|\underline{s}_{I1}) = D^{(1)}\underline{s}_{I1}$ and $E(\underline{s}|\underline{s}_{I2}) = D^{(2)}\underline{s}_{I2}$

$$D^{(1)} = \begin{bmatrix} I \\ 0 \\ D_1 \end{bmatrix} , \quad D^{(2)} = \begin{bmatrix} 0 \\ I \\ D_2 \end{bmatrix} . \qquad (3.2.45)$$

Finally, defining $H = T_{NJ}Q_{JJ}^{-1}T_{NJ}'$ with appropriately partitioned submatrices H_{ij} :

$$\langle V_{Cs_I Fd_J} | \mathcal{E} \rangle = -\frac{1}{2}\mathrm{trace}(D'HDC_{II})$$

$$= -\frac{1}{2}\mathrm{trace}\left\{ \begin{bmatrix} I & 0 & D_1' \\ 0 & I & D_2' \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \\ D_1 & D_2 \end{bmatrix} \begin{bmatrix} C_{I1I1} & 0 \\ 0 & C_{I2I2} \end{bmatrix} \right\}$$

$$= -\frac{1}{2}\text{trace}\begin{bmatrix}\left[H_{11}+H_{13}D_1+D_1'(H_{31}+H_{33}D_1)\right]C_{I1I1}, & \text{———} \\ \text{———}, & \left[H_{22}+H_{23}D_2+D_2'(H_{32}+H_{33}D_2)\right]C_{I2I2}\end{bmatrix},$$

(3.2.46)

$$\langle V_{Cs_{I1}Fd_J}|\mathcal{E}\rangle = -\frac{1}{2}\text{trace}\left\{D^{(1)'}HD^{(1)}C_{I1I1}\right\}$$

$$= -\frac{1}{2}\text{trace}\left\{\left[H_{11}+H_{13}D_1+D_1'(H_{31}+H_{33}D_1)\right]C_{I1I1}\right\}, \quad (3.2.47)$$

$$\langle V_{Cs_{I2}Fd_J}|\mathcal{E}\rangle = -\frac{1}{2}\text{trace}\left\{D^{(2)'}HD^{(2)}C_{I2I2}\right\}$$

$$= -\frac{1}{2}\text{trace}\left\{\left[H_{22}+H_{23}D_2+D_2'(H_{32}+H_{33}D_2)\right]C_{I2I2}\right\}. \quad (3.2.48)$$

(3.2.40) follows from a comparison of (3.2.46), (3.2.47), and (3.2.48). **QED.**

Following the example of Section 3.1, we say that decision vectors $\underline{d}_{J1}$ and $\underline{d}_{J2}$ do not interact if the value function may be expressed as

$$v(\underline{s};\underline{d}_{J1},\underline{d}_{J2},\underline{d}_{\bar{J}}) = v_1(\underline{s};\underline{d}_{J1},\underline{d}_{\bar{J}}) + v_2(\underline{s};\underline{d}_{J2},\underline{d}_{\bar{J}}). \quad (3.2.49)$$

<u>COROLLARY 4</u>: Suppose the decision variables composing the decision vector $\underline{d}_J$ for which flexibility is available may be partitioned into two vectors $\underline{d}_{J1}$ and $\underline{d}_{J2}$ that do not interact. Then

$$\langle V_{Cs_IFd_J}|\mathcal{E}\rangle = \langle V_{Cs_IFd_{J1}}|\mathcal{E}\rangle + \langle V_{Cs_IFd_{J2}}|\mathcal{E}\rangle. \quad (3.2.50)$$

<u>PROOF</u>: For convenience we assume decision variables are ordered so that

$$\underline{d} = \begin{bmatrix}\underline{d}_{J1} \\ \underline{d}_{J2} \\ \underline{d}_{\bar{J}}\end{bmatrix}. \quad (3.2.51)$$

For the quadratic value function, the non-interaction assumption means that the matrix Q_{JJ} has the diagonal form

$$Q_{JJ} = \begin{bmatrix} Q_{J1J1} & 0 \\ 0 & Q_{J2J2} \end{bmatrix} . \tag{3.2.52}$$

Then using (3.2.41),

$$\langle V_{Cs_I Fd_J} | \varepsilon \rangle = -\frac{1}{2}\text{trace}\left\{D'\left[T_{NJ1}, T_{NJ2}\right]\begin{bmatrix} Q^{-1}_{J1J1} & 0 \\ 0 & Q^{-1}_{J2J2} \end{bmatrix}\begin{bmatrix} T'_{NJ1} \\ T'_{NJ2} \end{bmatrix} D\, C_{II}\right\}$$

$$= -\frac{1}{2}\text{trace}(D'T_{NJ1}Q^{-1}_{J1J1}T'_{NJ1}DC_{II})$$

$$-\frac{1}{2}\text{trace}(D'T_{NJ2}Q^{-1}_{J2J2}T'_{NJ2}DC_{II})$$

$$= \langle V_{Cs_I Fd_{J1}} | \varepsilon \rangle + \langle V_{Cs_I Fd_{J2}} | \varepsilon \rangle . \tag{3.2.53}$$

QED.

CHAPTER 4

QUANTIZED INFORMATION AND DISCRETIZED DECISION SYSTEMS

In Section 2.3 the topic of evaluating imperfect information given partial flexibility was discussed. That discussion might have led the reader to presume that a decision maker would never prefer to use imperfect information or partial rather than complete flexibility in his decision model. Quite to the contrary, perfect information will typically be evaluated as if it were imperfect quantized information, and complete flexibility will usually be treated as if it were only partial discretized flexibility.

The reason for this is that quantization of information and discretization of decisions can provide the simplification necessary to reduce a complex decision problem to a computationally manageable size. The application of this principle is so common that it is likely to be taken for granted. Information is usually expressed numerically or otherwise categorized and, therefore, by necessity it is "rounded off." Similarly, our decisions are frequently constrained to be some multiple of a common number or unit: dozens, cases, minutes, etc. Quantization allows us to reduce an infinite number of possibilities to a finite number with which we find it easier to deal.

By the same principle, if the decision analyst treats his continuous or many-valued state variables as if they were roughly quantized, and his continuous or many-valued decision variables as if they were crudely discretized, he can frequently achieve an enormous simplification in his decision model. For this reason, information and flexibility will typically be accounted for by artificially quantizing state variables and artificially discretizing decision variables in a very rough manner.

The purpose of this chapter will be to investigate the consequences of introducing quantization into a decision model. The first sections address the issue of quantizing continuous valued state variables; the latter section, discretizing continuous decision variables. Our objective here is by no means meant to be a thorough investigation of quantization in decision analysis. Rather, we address the problem more as a necessary side issue of our study of the value of information-flexibility structures.

4.1 STATE VARIABLE QUANTIZATION

We begin by considering once again the basic decision model of Fig. 1.4. If the value of the state vector $\underline{s}$ was anticipated to become known to the decision maker prior to his setting of the decision vector $\underline{d}$, the utility of the decision problem (before $\underline{s}$ is revealed) and the optimum decision strategy are found by solving the functional optimization problem

$$\langle u|\mathcal{E}\rangle = \int_{\underline{s}\in S} \max_{\underline{d}\in D} \langle u|\underline{s},\underline{d},\mathcal{E}\rangle \{\underline{s}|\mathcal{E}\} \ . \tag{4.1.1}$$

Solution of (4.1.1) yields an optimal decision function $\underline{d}^*(\cdot)$ which maps the state space S into some subset of the action set D . Calculation of such a decision function is necessary in problems for which optimization after the receipt of information is impractical or in problems for which the setting of the decision vector $\underline{d}$ conditional upon $\underline{s}$ is part of the evaluation of a larger decision strategy.

For several reasons, however, the functional optimization indicated in (4.1.1) is rarely performed in practice. In many cases because of accounting problems, the precise value of $\underline{s}$ will not be reported to the decision maker. Rather its value will be rounded off to a more convenient number. In other situations the functional optimization required in (4.1.1) may be too difficult to perform; it will be far simpler to execute the optimization in a decision tree in which the distribution of $\underline{s}$ has been approximated by a probability mass function. Thus, we often find that practical matters lead us to quantize a random variable.

State variable quantization in decision models is usually accomplished by artificially concentrating the continuously distributed state variable probability mass on certain discrete points chosen within the state space S . This, however, causes some problems. After the decision strategy has been formulated and it comes time for the

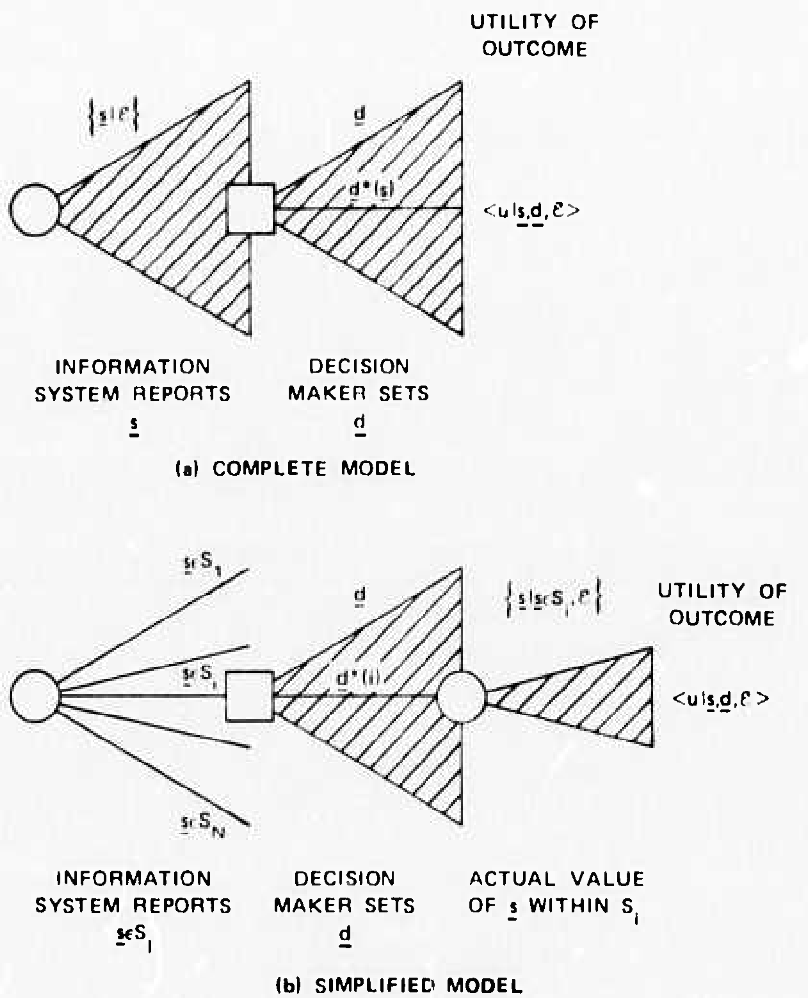


FIGURE 4.1 COMPLETE DECISION MODEL AND SIMPLIFICATION RESULTING FROM QUANTIZATION

decision maker to set $\underline{d}$ according to an optimal decision strategy $d^*(\cdot)$, the observed value of $\underline{s}$ may not even correspond to any of the discrete levels for which the decision rule was calculated. In such situations, the decision maker is likely to use the optimal decision setting corresponding to the "closest" quantized level. We formalize this last thought by imagining that a partition $P_{\underline{s}}^N = \left\{S_i\right\}_{i=1}^N$ has been formed on the set S of possible states of Nature. The action of the quantized information system is such that if $\underline{s} \in S_i$, the index i is communicated and made available for the decision process. Corresponding to each subset S_i the decision maker must find the decision setting $\underline{d}^*(i)$ that maximizes the expected utility given only that $\underline{s} \in S_i$.

The complete decision model and the simplification achieved through quantization are illustrated in Fig. 4.1. Notice that not only has the information system been simplified, but specification of the optimal decision strategy requires only the specification of the N vectors $\underline{d}^*(1),\ldots,\underline{d}^*(N)$, not the specification of a vector of functions $\underline{d}^*(\cdot)$ on S . Specifically, if $\underline{s} \in S_i$ the optimal decision setting maximizes the conditional expectation

$$\langle u|s \in S_i, d_i, \mathcal{E}\rangle = \int_{\underline{s} \in S_i} \langle u|\underline{s},\underline{d},\mathcal{E}\rangle \{\underline{s}|\underline{s} \in S_i, \mathcal{E}\} , \quad i=1,\ldots,N , \tag{4.1.2}$$

where

$$\{\underline{s}|\underline{s} \in S_i, \mathcal{E}\} = \frac{\{\underline{s}|\mathcal{E}\}}{\{\underline{s} \in S_i|\mathcal{E}\}} , \quad \underline{s} \in S_i \tag{4.1.3}$$

is the conditional distribution of $\underline{s}$ given that $\underline{s}$ lies in subset S_i and

$$\{\underline{s} \in S_i|\mathcal{E}\} = \int_{\underline{s} \in S_i} \{\underline{s}|\mathcal{E}\} \tag{4.1.4}$$

is the probability that $\underline{s}$ lies in S_i . Hence, to the partition $P_{\underline{s}}^N$ and given model $\left\{S,D,\{\underline{s}|\mathcal{E}\},v,u\right\}$ will correspond an expected utility

$$\langle u|P_{\underline{s}}^N,\mathcal{E}\rangle = \sum_{i=1}^{N} \{\underline{s} \in S_i|\mathcal{E}\} \langle u|\underline{s} \in S_i, \mathcal{E}\rangle = \sum_{i=1}^{N} \max_{d \in D} \int_{s \in S_i} \langle u|\underline{s},\underline{d},\mathcal{E}\rangle \{\underline{s}|\mathcal{E}\}. \tag{4.1.5}$$

EVALUATING A QUANTIZING SCHEME

Suppose that quantized information may be purchased at a price p_1. If $\langle u|\mathcal{E}\rangle$ is the expected utility given no information on $\underline{s}$, and $\langle u|P^N_{\underline{s}},p_1,\mathcal{E}\rangle$ is the expected utility given that the quantized information system is purchased at a price p_1, we define the price p_1 such that

$$\langle u|P^N_{\underline{s}},p_1,\mathcal{E}\rangle = \langle u|\mathcal{E}\rangle \tag{4.1.6}$$

to be the value of the partition $P^N_{\underline{s}}$ and denote this value by $\mathcal{V}(P^N_{\underline{s}})$. Similarly, suppose that complete (exact) information could be purchased at a price p_2, and let $\langle u|C\underline{s},p_2,\mathcal{E}\rangle$ denote the maximum expected utility if this purchase is made. We define the price p_2 such that

$$\langle u|C\underline{s},p_2,\mathcal{E}\rangle = \langle u|P^N_{\underline{s}},\mathcal{E}\rangle \tag{4.1.7}$$

as the loss associated with using the particular quantization scheme $P^N_{\underline{s}}$. We denote this quantity by $\mathcal{L}(P^N_{\underline{s}})$. $\mathcal{V}(P^N_{\underline{s}})$ may be interpreted as the maximum amount of money the decision maker would be willing to pay for the use of the quantized information system. We interpret $\mathcal{L}(P^N_{\underline{s}})$ as the economic loss a decision maker with a complete information system would have to sustain to make him indifferent between retaining it (the complete information system) and accepting instead the quantized system.

The number of subsets N comprising the partition $P^N_{\underline{s}}$ determines the level of quantization. Several possible levels of quantization are illustrated in Fig. 4.2. Notice that for $N = 1$ we have the situation of Fig. 2.1, and the loss of the quantized system is just the value of perfect information on $\underline{s}$ given perfect flexibility on $\underline{d}$. As N goes to infinity the quantized system approaches the complete model in which information and flexibility are available. Thus, the decision models with and without flexibility can be thought of as the two extremes of a collection of models which piece-wise approximate the optimal decision strategy over the state space.

Because calculation of the optimal strategy for the quantized model is, in principle, easier than that for the complete model, two

$$\langle u|P,\mathcal{E}\rangle$$

$$\max_{\underline{d}} \int_{\underline{s}\in S} \langle u|\underline{s},\underline{d},\mathcal{E}\rangle \{\underline{s}|\mathcal{E}\}$$

$$\max_{\underline{d}_1} \int_{\underline{s}\in S} \langle u|\underline{s},\underline{d},\mathcal{E}\rangle \{\underline{s}|\mathcal{E}\} + \max_{\underline{d}_2} \int_{\underline{s}\in S} \langle u|\underline{s},\underline{d},\mathcal{E}\rangle \{\underline{s}|\mathcal{E}\}$$

$$\sum_{i=1}^{N} \max_{\underline{d}_i} \int_{\underline{s}\in S_i} \langle u|\underline{s},\underline{d}_i,\mathcal{E}\rangle \{\underline{s}|\mathcal{E}\}$$

$$\int_{\underline{s}\in S} \max_{\underline{d}} \langle u|\underline{s},\underline{d},\mathcal{E}\rangle \{\underline{s}|\mathcal{E}\}$$

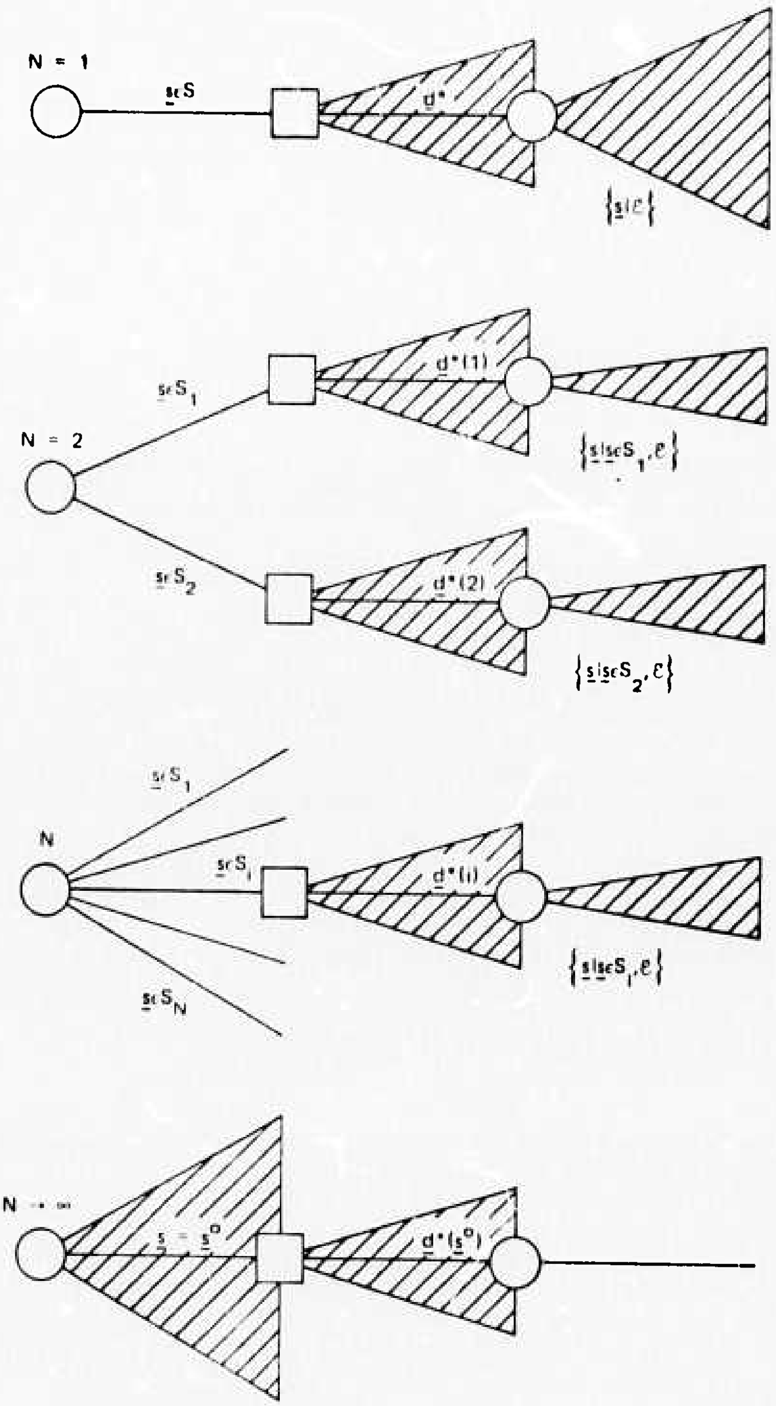


FIGURE 4.2 DECISION PROBLEMS FOR VARIOUS LEVELS OF QUANTIZATION N

important questions emerge: First, what is the value of a particular level of quantization? Second, given that the level N of quantization is fixed, how should the partition $P_{\underline{s}}^{N} = \left\{ S_i \right\}_{i=1}^{N}$ be determined? Equations (4.1.5), (4.1.6), and (4.1.7) move us an important step towards answering these questions. We can compare the relative advantage of using various quantizing schemes by comparing their expected utilities and associated values and losses. Fixing N , we can choose the quantizing partition $P_{\underline{s}}^{N} = \left\{ S_i \right\}_{i=1}^{N}$ so as to maximize this advantage. We apply these ideas to the entrepreneur's price and quantity decision in the example below.

4.2 QUANTIZING THE COST VARIABLE IN THE ENTREPRENEUR'S DECISION

Suppose for the entrepreneur's problem of Section 2.2 that information is available on the cost variable c . The decision maker must choose between using the complete model and the N-partition model of Fig. 4.3. We wish to determine the economic loss to be expected from using various levels of quantization, and we wish to design the quantizing partitions in such a way so as to minimize this loss.

TWO-LEVEL QUANTIZATION

The possible values for per unit production costs c are partitioned into two sets by the point y_1 . If $0 < c < y_1$, costs are "low" and an appropriate price p and quantity q are to be determined. If $y_1 < c < 1$, a price and quantity are chosen that are appropriate for "high" costs. The expected payoff using such a strategy is given by

$$\langle v|P_c^N,\mathcal{E}\rangle = \max_{p_1}\max_{q_1} \int_0^{y_1}_c \int_0^1_e \langle v|p,q,c,e,\mathcal{E}\rangle \{e|\mathcal{E}\}\{c|\mathcal{E}\}$$

$$+ \max_{p_2}\max_{q_2} \int_{y_1}^{1}_c \int_0^1_e \langle v|p,q,c,e,\mathcal{E}\rangle \{e|\mathcal{E}\}\{c|\mathcal{E}\}$$

$$= \max_{p_1}\max_{q_1}\left\{y_1\left[a+ba-\left(b+\frac{b^2}{2}+\frac{1}{2}\right)p-\frac{a^2}{2p}+\left(a-\frac{y_1}{2}\right)q-pbq-\frac{pq^2}{2}\right]\right\}$$

$$+ \max_{p_2}\max_{q_2}\left\{(1-y_1)\left[a+ba-\left(b+\frac{b^2}{2}+\frac{1}{2}\right)p-\frac{a^2}{2p}+\left(a-\frac{1+y_1}{2}\right)q-pbq-\frac{pq^2}{2}\right]\right\}. \tag{4.2.1}$$

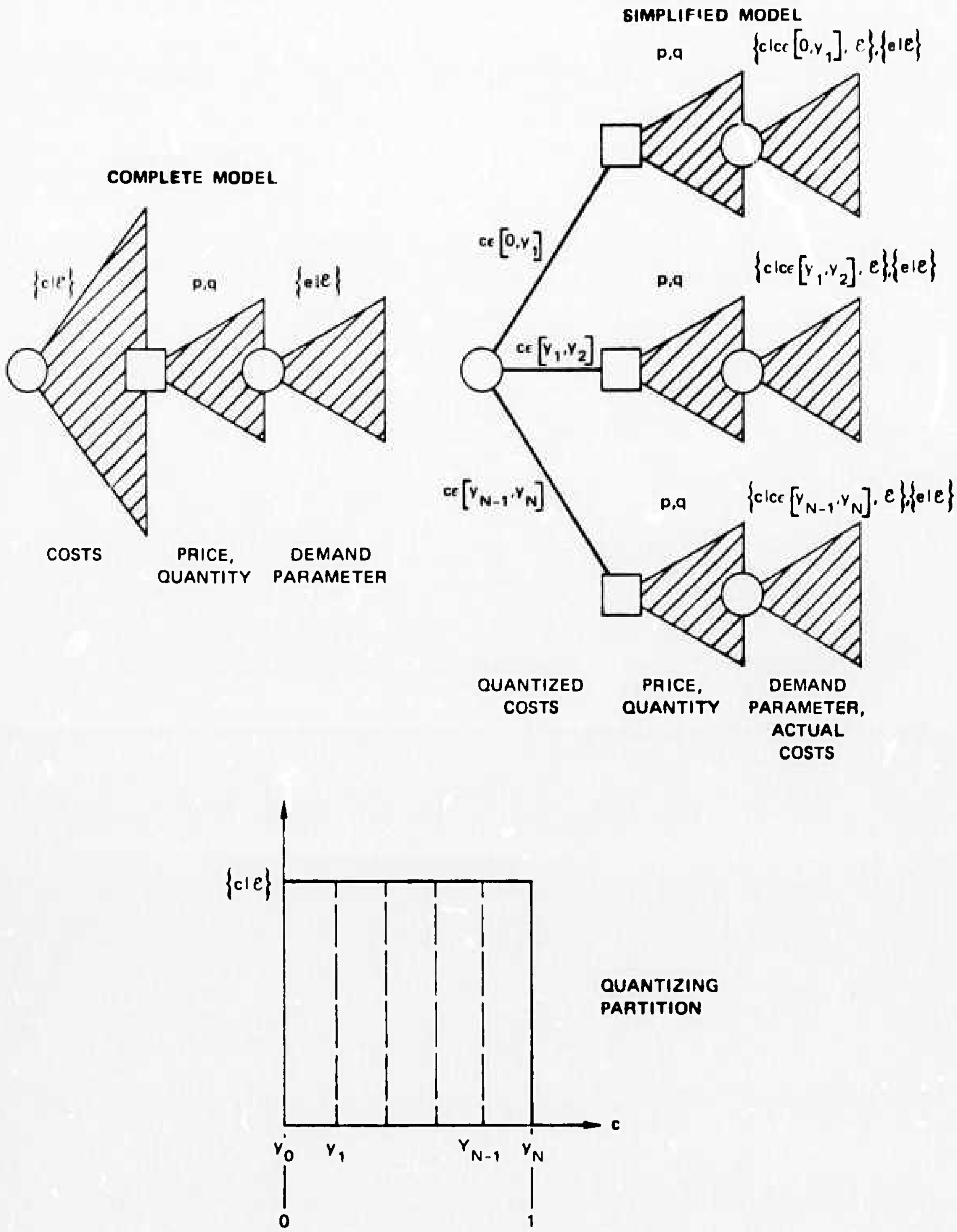


FIGURE 4.3 COMPLETE MODEL, SIMPLIFIED MODEL, AND QUANTIZING PARTITION FOR THE ENTREPRENEUR'S DECISION

Differentiating to perform the maximizations we obtain the decision rule

$$q_1^* = \frac{2a-y_1-2pb}{2p} \quad , \quad p_1^* = \sqrt{\frac{4ay_1-y_1^2}{8b+4}} \tag{4.2.2}$$

$$q_2^* = \frac{2a-(1+y_1)-2pb}{2p} \quad , \quad p_2{}^* = \sqrt{\frac{4a(1+y_1)-(1+y_1)^2}{8b+4}} \tag{4.2.3}$$

and a corresponding maximal expected payoff of

$$\langle v|P_c^2(y_1),\mathcal{E}\rangle = a + \frac{b}{2} - \frac{\sqrt{2b+1}}{2}\left[(4a-y_1)^{1/2}y_1^{3/2} + (4a-1-y_1)^{1/2}(1-y_1)\right] . \tag{4.2.4}$$

Notice that for $y_1 = 0$ or $y_1 = 1$, Eqs. (4.2.2), (4.2.3), and (4.2.4) reduce to Eqs. (B.4) and (B.5) of Appendix B. This makes sense when we interpret the latter equations as corresponding to a one-partition quantization of costs.

If the organizer sets $y_1 = 1/2$, the state space is partitioned in half and the maximum amount of information is conveyed by the quantized information system. In this case, plugging $y_1 = .5$, $a = 2.25$, and $b = .5$ into (4.2.4) gives

$$\langle v|P_c^2,\mathcal{E}\rangle = \$585,277 \ , \tag{4.2.5}$$

which is \$85,139 better than if no information on costs is provided, and \$54,139 short of the value produced with the complete information system. However, from (4.2.4) we see that the best two-state information system is obtained by choosing y_1 so as to minimize

$$\begin{aligned}(4a-y_1)^{1/2}y_1^{3/2} &+ (4a-1-y_1)^{1/2}(1+y_1)^{1/2}(1-y_1) \\ &= (9-y_1)^{1/2}y_1^{3/2} + (8-y_1)^{1/2}(1+y_1)^{1/2}(1-y_1) \ .\end{aligned} \tag{4.2.6}$$

The value of y_1 that minimizes this expression is approximately

$$y_1 = 0.35165 \ . \tag{4.2.7}$$

Thus, the best two-level quantizing scheme for a strategy contingent only on the information "costs are high" or "costs are low" would be achieved by defining costs as "high" or "low" according to whether or not they exceed \$351.65 per unit. Plugging (4.2.7) into (4.2.4) we obtain the maximum expected payoff if a two-partition information system is used for costs:

$$\langle v|P_c^2,\mathcal{E}\rangle_{max} = \$592{,}328 \;. \tag{4.2.8}$$

N-LEVEL QUANTIZATION

Define $y_0 = 0$ and $y_N = 1$ and assume that the information system quantizes the cost variable c into N intervals the i^{th} of which is bounded by the points y_{i-1} and y_i. If the information system reports only the interval into which costs fall, the expected payoff will be

$$\langle v|P_c^N(y_1,\ldots,y_{N-1}),\mathcal{E}\rangle = \sum_{i=1}^{N} \max_{p_i} \max_{q_i} \int_{c=y_{i-1}}^{y_i} \int_{e=0}^{1} \langle v|p,q,c,e,\mathcal{E}\rangle\{e|\mathcal{E}\}\{c|\mathcal{E}\} \,. \tag{4.2.9}$$

Differentiating as in the two-level case we obtain expressions for the optimal prices and quantities and the corresponding expected payoff.

$$p_i^* = \sqrt{\frac{4a(y_i+y_{i+1})-(y_i+y_{i+1})^2}{8b+4}} \;, \quad q_i^* = \frac{2a-(y_i+y_{i+1})}{2p_i^*} - b \;, \tag{4.2.10}$$

$$\langle v|P_c^N(y_1,\ldots,y_{N-1}),\mathcal{E}\rangle$$

$$= a+\frac{b}{2} - \frac{\sqrt{2b+1}}{2} \sum_{i=1}^{N} \left[(4a-y_{i-1}-y_i)^{1/2}(y_i+y_{i-1})^{1/2}(y_i-y_{i-1})\right] . \tag{4.2.11}$$

Due to the assumed linearity of the decision maker's utility function, the economic loss from using such a quantizing scheme is

$$\mathcal{L}(P_c^N) = \langle v|Cc,\mathcal{E}\rangle - \langle v|P_c^N(y_1,\ldots,y_{N-1}),\mathcal{E}\rangle \,. \tag{4.2.12}$$

Table 4.1 consists of numerical evaluations of (4.2.10) and (4.2.12). The calculations were made with $a = 2.25$, $b = .5$ for equal interval partitions and for the optimal quantizing partitions for several levels of quantization N. Figure 4.4 shows a plot of quantizing loss versus the number of quantizing levels N for the two quantizing schemes. Notice that quantizing loss falls roughly as N^{-2}.

4.3 QUANTIZATION IN THE TWO-VARIABLE QUADRATIC PROBLEM

In this section we shall explore the effects of information quantization on a two-variable quadratic decision problem of the form introduced in Chapter 3. For such decision problems we are able to determine necessary conditions that an optimal quantizing partition must satisfy and to evaluate the effects on quantizing loss of using various levels of quantization and various suboptimal quantizing schemes. The results are of interest because, as stated in Chapter 3, the quadratic function is frequently a good approximation to a complex value function.

Consider the simplest quadratic decision problem in which there is a single state variable s and a single decision variable d. We assume that the decision space D is the real line R and the state space S is a segment of R. The value function is assumed quadratic in s and d and for every s to attain a unique maximum with respect to d. Then, according to the argument of Section 3.2, by suitably defining origins there is no loss in generality in assuming

$$v(s,d) = tsd + \frac{1}{2} qd^2 \tag{4.3.1}$$

with $E(s) = 0$, $\mathrm{Var}(s) = \sigma^2$, and $q < 0$.

Our decision maker is assumed to be risk indifferent so that basing his decisions on expected payoff is equivalent to the criteria of expected utility. We suppose that, for the aid of the decision maker, a quantized information system has been instituted which induces a partition $P_s^N = \left\{S_i\right\}_{i=1}^N$ on the region S. The maximum expected payoff which can be attained using this information system is

TABLE 4.1

Loss and Decision Strategy for Various Quantized Information Systems for Costs in the Entrepreneur's Decision

QUANTIZING LEVEL N	QUANTIZING SCHEME & PARTITION BOUNDARIES	DECISION RULE price in thousands of dollars - quantity in thousands of units	QUANTIZING LOSS $\ell(P_c^N)$
1	—	$p^*=1.000$ $q^*=1.250$	\$139,416
2	Equal Interval .5000	$p_1^*=.728$ $q_1^*=2.243$ $p_2^*=1.186$ $q_2^*=.765$	\$54,139
2	Optimal .3517	$p_1^*=.617$ $q_1^*=2.864$ $p_2^*=1.137$ $q_2^*=.885$	\$47,088
3	Equal Interval .3333 .6667	$p_1^*=.601$ $q_1^*=2.966$ $p_2^*=1.000$ $q_2^*=1.250$ $p_3^*=1.236$ $q_3^*=.646$	\$30,722
3	Optimal .1783 .5207	$p_1^*=.443$ $q_1^*=4.378$ $p_2^*=.852$ $q_2^*=1.732$ $p_3^*=1.192$ $q_3^*=.749$	\$23,739
4	Equal Interval .2500 .5000 .7500		\$20,482
4	Optimal .1075 .3175 .6198		\$14,367
5	Equal Interval .2000 .4000 .6000 .8000		\$14,942

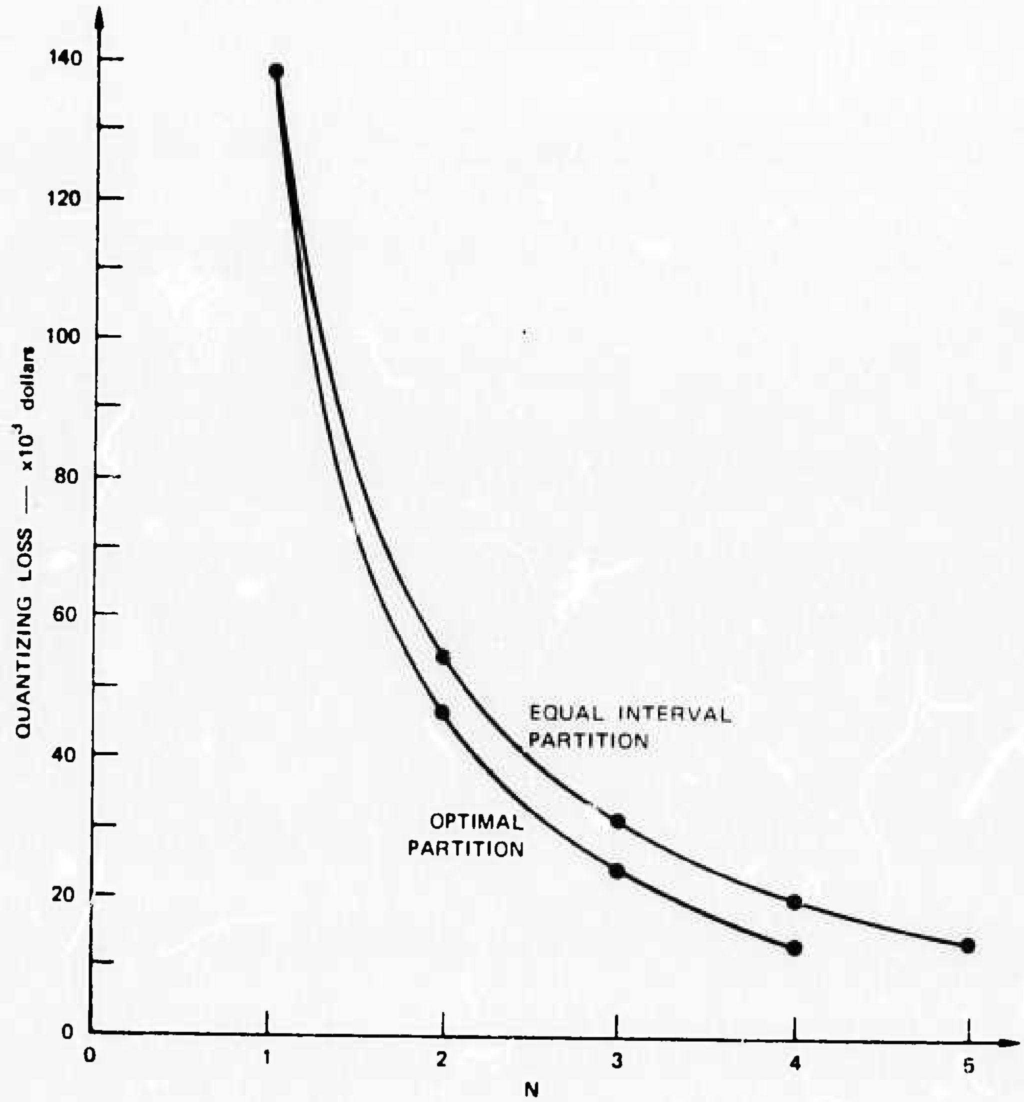


FIGURE 4.4 QUANTIZING LOSS VERSUS NUMBER OF QUANTIZING LEVELS IN THE ENTREPRENEUR'S DECISION

$$\langle v|P_s^N,\mathcal{E}\rangle = \underset{i}{E}\left\{\max_{d_i} E[v(s,d)\,|\,s\in S_i]\right\}$$

$$= \sum_{i=1}^{N} P_i \max_{d_i}[td_i\, E(s\,|\,s\in S_i) + \frac{1}{2}\, qd_i^2] \ , \qquad (4.3.2)$$

where P_i is the probability that s falls in S_i and

$$\sum_{i=1}^{N} P_i = 1 \ .$$

Differentiating in order to find the maximizing d_i , we obtain the decision variable setting the decision maker ought to use if he learns that s falls in S_i ,

$$d_i^* = -\frac{t}{q}E(s\,|\,s\in S_i),\ i = 1,\dots,N \ . \qquad (4.3.3)$$

Substituting this into (4.3.2),

$$\langle v|P_s^N,\mathcal{E}\rangle = -\frac{t^2}{2q}\sum_{i=1}^{N} P_i E(s\,|\,s\in S_i)^2 \ . \qquad (4.3.4)$$

Defining V_i as the conditional variance of s given that s falls in S_i ,

$$V_i = E(s^2\,|\,s\in S_i) - E(s\,|\,s\in S_i)^2 \ , \qquad (4.3.5)$$

we have

$$\langle v|P_s^N,\mathcal{E}\rangle = -\frac{t^2}{2q}\sum_{i=1}^{N} P_i[E(s^2\,|\,s\in S_i) - V_i]$$

$$= -\frac{t^2}{2q}\,\sigma^2\left(1 - \frac{1}{\sigma^2}\sum_{i=1}^{N} P_i V_i\right) . \qquad (4.3.6)$$

From (3.2.39) we see that the term in front of the expression in the parentheses is just the value of clairvoyance on s:

$$\mathcal{V}(Cs) = -\frac{t^2}{2q}\sigma^2 . \tag{4.3.7}$$

Finally, since the expected payoff given no information on s is zero, the value of the quantizing system equals the expected payoff:

$$\mathcal{V}(P_s^N) = \mathcal{V}(Cs)(1-L_1) , \tag{4.3.8}$$

where

$$L_1 = \frac{1}{\sigma^2}\sum_{i=1}^{N} P_i V_i . \tag{4.3.9}$$

In view of the above, we see that choosing the quantizing system so as to maximize expected payoff is equivalent to solving the extremal problem defined by

$$\inf_{\bigcup_{i=1}^{N} S_i = S} \sum_{i=1}^{N} P_i V_i . \tag{4.3.10}$$

Equivalent forms of problem (4.3.10) have appeared in several papers dealing with grouping problems with different objectives of the grouping [2,3,4].

OPTIMAL QUANTIZING PARTITION

It is a simple matter to derive the optimal quantizing partition if we return to Eq. (4.3.4). From (4.3.10) it should be intuitively clear (and it is easy to prove) that the optimal partition consists of a breakdown of the space S into connected intervals. Therefore, let the regions S_i be the segments (y_{i-1}, y_i) with $y_0 \leq y_1 \leq \dots \leq y_N$, where y_0 and y_N are respectively the (possibly infinite) initial and final points of the region S. Then if $f(s) = \{s|\mathcal{E}\}$ is the density function describing the distribution of s,

$$\langle v|P_s^N,\mathcal{E}\rangle = -\frac{t^2}{2q}\sum_{i=1}^{N}\frac{A_i^2}{P_i} , \tag{4.3.11}$$

where

$$A_i = \int_{y_{i-1}}^{y_i} s\, f(s)ds \ . \tag{4.3.12}$$

If we wish to maximize (4.3.11) for fixed N, we get necessary conditions by differentiating with respect to the y_i and setting derivatives equal to zero. The terms of the sum involving y_i are

$$\frac{A_i^2}{P_i} + \frac{A_{i+1}^2}{P_{i+1}} \ . \tag{4.3.13}$$

Differentiating,

$$\frac{\partial}{\partial y_i}\left[\frac{A_i^2}{P_i} \quad \frac{A_{i+1}^2}{P_{i+1}}\right] = \frac{2A_i}{P_i}\frac{\partial A_i}{\partial y_i} + \frac{2A_{i+1}}{P_i}\frac{\partial A_{i+1}}{\partial y_i} - \frac{A_i^2}{P_i^2}\frac{\partial P_i}{\partial y_i} - \frac{A_{i+1}^2}{P_{i+1}^2}\frac{\partial P_{i+1}}{\partial y_i}$$

$$= f(y_i)\left(\frac{A_i}{P_i} - \frac{A_{i+1}}{P_{i+1}}\right)\left[2y_i - \left(\frac{A_i}{P_i} - \frac{A_{i+1}}{P_{i+1}}\right)\right] = 0 \ . \tag{4.3.14}$$

Noting that

$$\frac{A_i}{P_i} = \langle s \mid y_{i-1} < s < y_i, \mathcal{E}\rangle \ , \tag{4.3.15}$$

then, since f is nonzero on S and the optimal partition consists of contiguous intervals, (4.3.14) implies that

$$y_i = \frac{1}{2}(\langle s \mid y_{i-1} < s < y_i, \mathcal{E}\rangle + \langle s \mid y_i < s < y_{i+1}, \mathcal{E}\rangle) \ . \tag{4.3.16}$$

Thus for the two-variable quadratic decision problem, the optimal partition is independent of the parameters of the value function and is such that partition boundaries are located half-way between the conditional means of the neighboring partitions.

If f is unimodal the solution to (4.3.16) will be unique. Otherwise, all the critical points must be calculated in order to identify the partition that maximizes (4.3.11). Except for trivial distributions, solutions to (4.3.16) will have to be determined using some sort of iterative technique. Table 4.2 gives the optimal partition

TABLE 4.2

Optimal Quantizing Partitions for the Quadratic Problem

NUMBER OF LEVELS N	UNIFORM DISTRIBUTION $f(s) = \frac{1}{2}, -1 \le s \le 1$		EXPONENTIAL DISTRIBUTION $f(s) = e^{-s}, s \ge 0$		NORMAL DISTRIBUTION $f(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} \; -\infty < s < \infty$	
	PARTITION BOUNDARIES	PARTITION PROBABILITIES	PARTITION BOUNDARIES	PARTITION PROBABILITIES	PARTITION BOUNDARIES	PARTITION PROBABILITIES
2	.000	.500 .500	1.594	.797 .203	.000	.500 .500
3	-.333 .333	.333 .333 .333	1.018 2.611	.639 .288 .073	-.612 .612	.270 .459 .270
4	-.500 .000 .500	.250 .250 .250 .250	.754 1.722 3.365	.530 .300 .135 .035	-.981 .000 .981	.164 .336 .336 .164
5	-.600 -.200 .200 .600	.200 .200 .200 .200 .200	.600 1.354 2.372 3.966	.451 .291 .165 .074 .091	-1.230 -.395 .395 1.230	.109 .237 .307 .237 .109
6	-.667 -.333 .000 .333 .667	.167 .167 .167 .167 .167 .167	.499 1.100 1.854 2.872 4.455	.393 .274 .176 .100 .045 .012	-1.449 -.660 .000 .660 1.449	.074 .181 .245 .245 .181 .074

boundaries for the normalized uniform, exponential and normal distributions. The normal results were obtained in another context by Cox [3]. Data for the other distributions were calculated from the results of the following two examples.

<u>Example</u>: <u>Uniform Distribution</u>

Suppose that s is uniformly distributed. Then

$$<s \mid y_{i-1} \le s < y_i, \mathcal{E}> = \frac{y_{i-1} + y_i}{2} \quad . \tag{4.3.17}$$

Substituting into (4.3.16) and solving for y_i, we get

$$y_i = \frac{y_{i-1} + y_{i+1}}{2} \quad . \tag{4.3.18}$$

Therefore, the optimal partition for the uniform distribution is obtained by equally spacing the y_i .

<u>Example</u>: <u>Exponential Distribution</u>

For the exponential distribution

$$f(s) = e^{-s} \quad , \quad s \ge 0 \quad , \tag{4.3.19}$$

we have

$$<s \mid y_{i-1} \le s < y_i, \mathcal{E}> = \frac{\int_{y_{i-1}}^{y_i} s\, e^{-s} ds}{\int_{y_{i-1}}^{y_i} e^{-s}\, ds} = 1 + \frac{y_i e^{-y_i} - y_{i-1} e^{-y_{i-1}}}{e^{-y_i} - e^{-y_{i-1}}} \quad . \tag{4.3.20}$$

Substituting into (4.3.16),

$$y_i = \frac{1}{2}\left[2 + \frac{y_i e^{-y_i} - y_{i-1} e^{-y_{i-1}}}{e^{-y_i} - e^{-y_{i-1}}} + \frac{y_{i+1} e^{-y_{i+1}} - y_i e^{-y_i}}{e^{-y_{i+1}} - e^{-y_i}}\right] . \tag{4.3.21}$$

With a little algebra this may be expressed as

$$\frac{y_i - y_{i-1}}{2} = \left[1 - e^{-(y_i - y_{i-1})}\right]\left[1 + \frac{y_{i+1} - y_i}{2} \cdot \frac{1}{1 - e^{-(y_i - y_{i+1})}}\right] . \tag{4.3.22}$$

Consider first $i = N-1$ and observe that $y_N = \infty$. Then, (4.3.22) becomes

$$\frac{y_{N-1} - y_{N-2}}{2} = \left[1 - e^{-(y_{N-1} - y_{N-2})}\right] . \tag{4.3.23}$$

Plugging $i = N-2$ into (4.3.22) and using (4.3.23),

$$\frac{y_{N-2} - y_{N-3}}{2} = \left[1 - e^{-(y_{N-2} - y_{N-3})}\right]\left[1 + \frac{1 - e^{-(y_{N-1} - y_{N-2})}}{1 - e^{-(y_{N-2} - y_{N-1})}}\right]$$

$$= \left[1 - e^{-(y_{N-2} - y_{N-3})}\right] \frac{y_{N-1} - y_{N-2}}{2} . \tag{4.3.24}$$

Using (4.3.22), (4.3.23), and (4.3.24) we can calculate the optimal y_i for various values of N . For example, if $N = 3$, (4.3.23) becomes

$$\frac{y_2 - y_1}{2} = \left[1 - e^{-(y_2 - y_1)}\right] , \tag{4.3.25}$$

which has the solution $y_2 - y_1 = 1.5936$. Then using (4.3.24) and the fact that $y_0 = 0$,

$$\frac{y_1}{2} = (1 - e^{-y_1}) \frac{1.5936}{2} , \tag{4.3.26}$$

which yields $y_1 = 1.0176$, $y_2 = 2.6112$.

DEPENDENCE OF QUANTIZING LOSS ON QUANTIZATION SCHEME

From (4.3.8) and the assumed linearity of our decision maker's utility function, the quantizing loss associated with the partition $P_s^N = \left\{ S_i \right\}_{i=1}^N$ is given by

$$\mathcal{L}(P_s^N) = U'(Cs) \cdot L_1 . \tag{4.3.27}$$

The quantity L_1 defined by (4.3.9) is a convenient measure of the losses introduced into the problem due to quantization. It is of interest to ascertain the effect of the number of levels N and of departures from the optimal partitions on quantizing loss. Consider the case of a uniformly distributed s. Using (4.3.18) in (4.3.9)

$$L_1 = \frac{1}{\sigma^2} \sum_{i=1}^{N} P_i V_i = \frac{1}{\sigma^2} \sum_{i=1}^{N} \frac{1}{N} \cdot \frac{\sigma^2}{N^2} = \frac{1}{N^2} \, . \tag{4.3.28}$$

We should expect approximately this sort of dependence in general. In fact if N is relatively large the probability density does not vary much from one end of a partition to the other and is well approximated by a density function whose magnitude over each partition is a constant equal to the mean of its values at the partition end points. If this is the case, the best way of partitioning to a level $2N$ is to divide each partition in half; it is easy to see that this will result in a value of L_1 for $2N$ partitions, which is $1/4$ the value for N partitions.

In Table 4.3 values of L_1 are given for the three distributions under their respective optimal partitions and under equiprobable partitions. Notice that, although the optimal partitions for the exponential and normal distributions differ considerably from equiprobable partitions, the latter perform well, indicating that the function L_1 is quite flat in the neighborhood of its minimum. This is a fortunate result as it implies that the optimal quantizing partition need not be specified too precisely. A plot of the minimum L_1 versus N appears in Fig. 4.5. Notice that L_1 decreases rapidly up to about $N = 3$. Beyond three quantizing levels a unit increase in N causes a relatively small decrease in L_1. This tells us that most of the value of information can be obtained by using rough quantization to as few as three levels.

A GRAPHICAL TECHNIQUE FOR DETERMINING THE OPTIMAL QUANTIZING PARTITION

We discuss here a graphical method for obtaining the optimal quantizing partition for the two-variable quadratic problem given an arbitrary prior distribution f.

At the heart of our method lies a technique for quickly and accurately approximating the mean $\bar{x}$ of a real valued continuous

TABLE 4.3

Values of the Quantizing Loss Factor L_1

DISTRIBUTION	PARTITION	N=2	3	4	5	6
Uniform	Optimal	.2500	.1111	.0625	.0400	.0278
	Equiprobable	.2500	.1111	.0625	.0400	.0278
Exponential	Optimal	.3524	.1797	.1090	.0731	.0522
	Equiprobable	.5195	.3509	.2649	.2127	.1777
Normal	Optimal	.3634	.1902	.1175	.0798	.0580
	Equiprobale	.3634	.2068	.1387	.1031	.0806

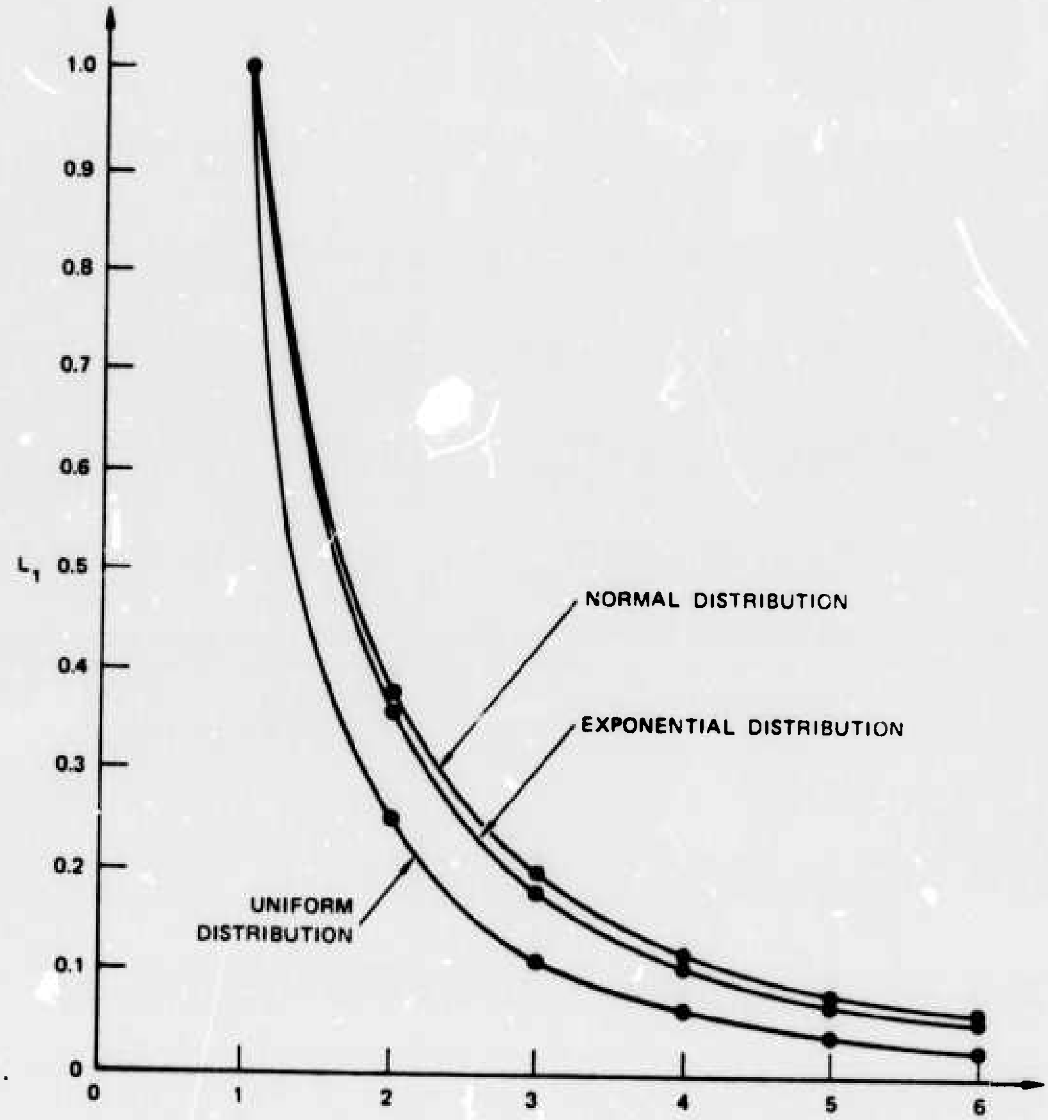


FIGURE 4.5 OPTIMUM QUANTIZING LOSS FACTOR VERSUS NUMBER OF QUANTIZING LEVELS FOR THE QUADRATIC PROBLEM

random variable x . We suppose that x is described by a cumulative distribution function G and corresponding probability density function g which is zero outside some (possible infinite) range (a,b) . Then

$$\bar{x} = \int_0^b xg(x)\,dx + \int_a^0 xg(x)\,dx . \tag{4.3.29}$$

Substituting

$$x = \int_0^x d\alpha = -\int_x^0 d\alpha , \tag{4.3.30}$$

we can replace the single integrals by double integrals,

$$\bar{x} = \int_0^b \int_0^x g(x)\,d\alpha dx - \int_a^0 \int_x^0 g(x)\,d\alpha dx . \tag{4.3.31}$$

Reversing the order of integration,

$$\bar{x} = \int_0^b \int_\alpha^b g(x)\,dxd\alpha - \int_a^0 \int_a^\alpha g(x)\,dxd\alpha$$

$$= \int_0^b [1 - G(\alpha)]\,d\alpha - \int_a^0 G(\alpha)\,d\alpha . \tag{4.3.32}$$

Equation (4.3.32) states that the mean is equal to the area lying between the cumulative distribution function and unity in the range $(0,b)$ less the area under the cumulative distribution function in the range $(a,0)$. If we now translate the axis to the right a distance d --that is, we define a random variable $x' = x-d$ --and choose d so that the difference between the areas is zero, we must have $\bar{x}' = \bar{x} - d = 0$, or $d = \bar{x}$. This shows that the mean of any real valued continuous random variable x is that point on the x axis for which a vertical line through the point is such that the area to the left under the cumulative distribution function is equal to the area to the right that is between the cumulative distribution function and unity.

Return now to the problem of approximating the optimal quantizing partition. Working on a plot of the cumulative distribution function, we begin by guessing a value for y_1. The conditional mean of the region partitioned by (y_0, y_1), which we denote in Fig. 4.6 by x_1, is now estimated using our graphical technique. When x_1 is properly positioned at the conditional mean, the areas of the shaded regions R_1 and R_1' will be equal.

Next, a distance $y_1 - x_1$ is marked off to the right of the point y_1, and the resulting point is called x_2. If y_2 is placed so as to satisfy (4.3.16), x_2 will be the conditional mean of the region (y_1, y_2), and we can use the fact that the areas R_2 and R_2' must be equal to aid us in placing y_2. Continuing with this method, all the boundary points through y_{N-1} may be placed. Finally, if y_1 was chosen correctly, then the point x_{N-1} will be at the conditional mean of the region (y_{N-1}, y_N). If x_{N-1} is too far to the left [right], then the process must be repeated using a larger [smaller] value for y_1.

4.4 DECISION VARIABLE DISCRETIZATION

Just as practical matters often lead to a quantization of the state space, they may also require us to restrict the feasible decision set to a class representable by a finite set of parameters. Again the reasons for this are primarily technical. Numerical expression of an optimal decision strategy inherently results in discretization. Also, the optimization can frequently be greatly simplified if it is performed in a decision tree which limits the possible decision settings to some small finite number. In fact, use of a digital computer by its very nature requires discretization of all continuous variables.

We are led, therefore, to the problem of choosing the class of alternatives to which our solution shall be restricted. As with state variable quantization, the key point to bear in mind is that the reduction in complexity to be obtained through rough discretization of the feasible decision space is obtained at a price of a reduction in performance. Let us pursue this topic by once again considering the two-state quadratic problem.

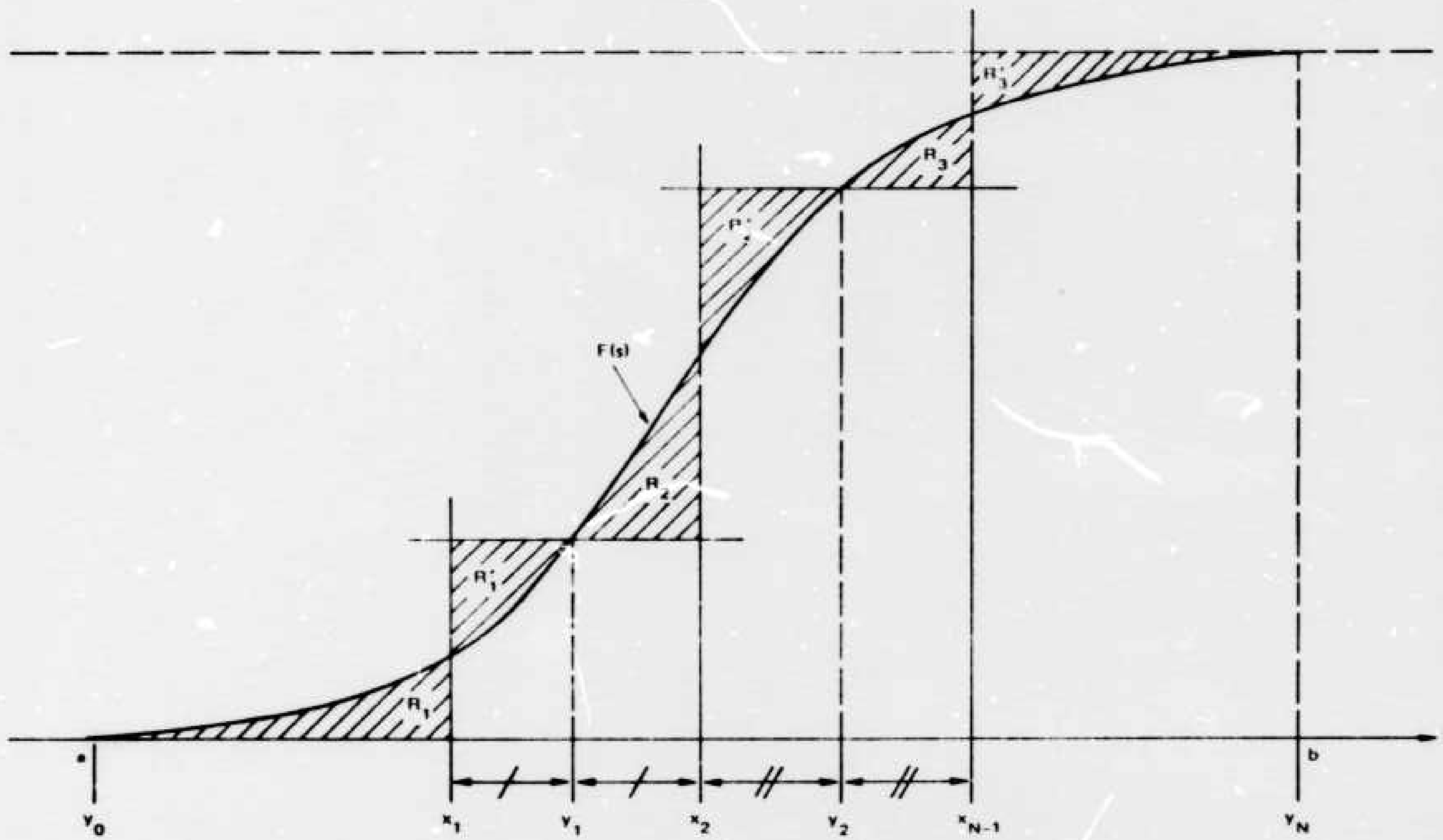


FIGURE 4.6 GRAPHICAL PROCEDURE FOR OBTAINING THE OPTIMAL QUANTIZING PARTITION FOR THE QUADRATIC PROBLEM

4.5 DISCRETIZATION IN THE TWO VARIABLE QUADRATIC PROBLEM

We assume that our decision maker is faced with the two-variable quadratic decision problem described in Section 4.3. His state of information is $\mathcal{S}$. We wish to know what economic loss will result if the decision maker uses the simplified model of Fig. 4.7 in which the infinite decision space D of the complete model is replaced by the set of points $\{d_1, \ldots, d_M\}$.

Let d^* denote the optimal decision setting for the complete model and let $D_d^M = \{d_i\}_{i=1}^{M}$ be the discretization of the decision space. Any discretization which contains d^* will, of course, result in zero loss. However, we presume that at this point in the modelling process d^* is not precisely known. Let $\ell(D_d^M)$ denote the expected loss in payoff corresponding to the discretization D_d^M . Then, by expansion,

$$\ell(D_d^M) = \sum_{i=1}^{M} \begin{bmatrix} \text{Probability } d_i \text{ is} \\ \text{chosen optimal by} \\ \text{simplified model} \end{bmatrix} \times \begin{bmatrix} \text{Expected loss if } d_i \\ \text{is chosen optimal} \\ \text{by simplified model} \end{bmatrix} . \quad (4.5.1)$$

In the present case, the maximum expected payoff

$$\max_d \langle v|d,\mathcal{S}\rangle = -\frac{1}{2}\frac{t^2}{q} E(s|\mathcal{S})^2 \qquad (4.5.2)$$

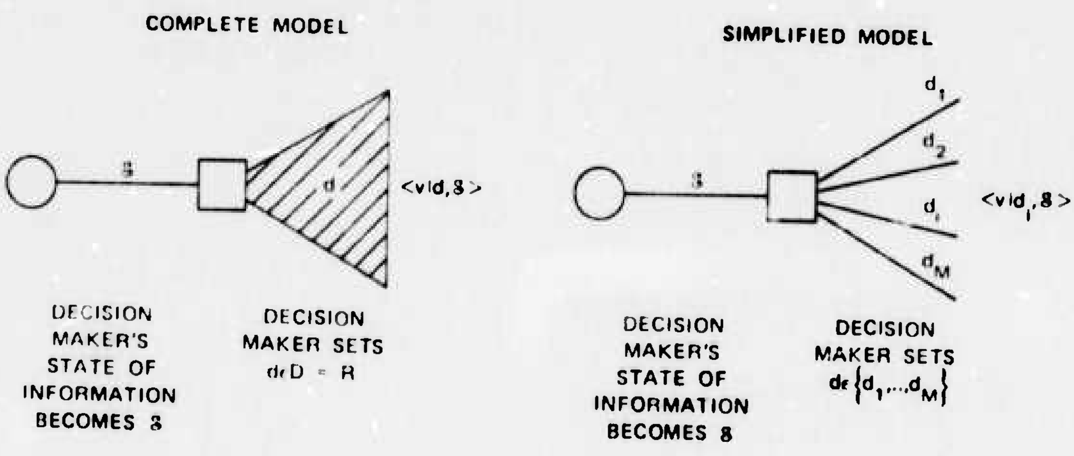


FIGURE 4.7 COMPLETE DECISION MODEL AND SIMPLIFICATION RESULTING FROM DISCRETIZATION

is achieved with

$$d^* = -\frac{t}{q} E(s|\mathcal{S}) \quad . \tag{4.5.3}$$

Defining $d' = d - d^*$ as the deviation from the optimal decision setting,

$$\langle v|d=d'+d^*,\mathcal{S}\rangle = -\frac{1}{2}\frac{t^2}{q} E(s|\mathcal{S})^2 + \frac{1}{2} qd'^2 \quad . \tag{4.5.4}$$

Thus, the expected loss in payoff if the decision variable is set to d_i is

$$\langle v|d^*,\mathcal{S}\rangle - \langle v|d_i,\mathcal{S}\rangle = -\frac{1}{2} q(d_i - d^*)^2 \quad . \tag{4.5.5}$$

Now let us explore the behavior of the simplified model. If $\left\{d_i\right\}_{i=1}^{M}$ is the existing discretization, which setting d_i will the simplified model choose as optimal? Suppose again that d^* is the optimal decision in the complete model. We will determine the value of d^*, say x_i, for which the simplified model is indifferent between alternatives d_i and d_{i+1}. Setting

$$\langle v|d_i,\mathcal{S}\rangle = \langle v|d_{i+1},\mathcal{S}\rangle \quad , \tag{4.5.6}$$

or using (4.5.5),

$$-\frac{1}{2} q(d_i - d^*)^2 = -\frac{1}{2} q(d_{i+1} - d^*)^2 \quad , \tag{4.5.7}$$

we obtain

$$d^* = x_i = \frac{d_i + d_{i+1}}{2} \quad . \tag{4.5.8}$$

Thus, the simplified quadratic model chooses the discretized decision setting that lies nearest the true optimal value.

We may now write mathematical expressions for each term on the right-hand side of (4.5.1). We assume that the decision analyst has through some means derived a prior distribution $\{d^*|\mathcal{S}\}$ on the optimal decision setting d^* for the complete model. Then the a priori probability that decision d_i will be chosen optimal by the simplified model is

$$P_i = \int_{\frac{d_{i-1}+d_i}{2}}^{\frac{d_i+d_{i+1}}{2}} \{d^*|\mathcal{S}\} \ , \quad i = 1,\ldots,M \ , \tag{4.5.9}$$

and the expected loss given that d_i is chosen optimal is

$$\frac{-q}{2P_i} \int_{\frac{d_{i-1}+d_i}{2}}^{\frac{d_i+d_{i+1}}{2}} (d_i - d^*)^2 \{d^*|\mathcal{S}\} \ , \quad i = 1,\ldots,M \ , \tag{4.5.10}$$

where d_0 and d_{M+1} are defined such that $(d_0+d_1)/2 = a$, $(d_M+d_{M+1})/2 = b$ and (a,b) is the range over which $\{d^*|\mathcal{S}\}$ is defined. Using the above, (4.5.1) becomes

$$\ell(D_d^M) = -\frac{q}{2} \sum_{i=1}^{M} \int_{\frac{d_{i-1}+d_i}{2}}^{\frac{d_i+d_{i+1}}{2}} (d_i - d^*)^2 \{d^*|\mathcal{S}\} \ . \tag{4.5.11}$$

A SPECIAL CASE

Before turning to the question of how the discretization $\left\{d_i\right\}_{i=1}^{M}$ should be chosen, we shall present a special case that appears to be an analog to the quantization problem of Section 4.3. Suppose that the uncertainty in d^* arises because the particular value that s obtains is not yet known. This would be the case, for example, if the discretized model were being designed to be used repeatedly for a number of independent trials. Then, since for the complete model

$$d^* = -\frac{t}{q} s \ , \tag{4.5.12}$$

if $\mathcal{S} = \mathcal{E}$ and $\{s|\mathcal{E}\} = f(s)$,

$$\{d^*|\mathcal{S}\} = \frac{q}{t} f(-\frac{q}{t} d^*) \ . \tag{4.5.13}$$

In this case, (4.5.11) becomes

$$\mathcal{L}(D_d^M) = \frac{q^2}{2t} \sum_{i=1}^{M} \int_{\frac{d_{i-1}+d_i}{2}}^{\frac{d_i+d_{i+1}}{2}} (d_i - d^*)^2 \, f(-\frac{q}{t} d^*) \, dd^* \, . \qquad (4.5.14)$$

Defining the change of variable $s^* = -qd^*/t$ and letting $s_i = -qd_i/t$, this may be written

$$\mathcal{L}(D_d^M) = -\frac{t^2}{2q} \sum_{i=1}^{M} \int_{\frac{s_{i-1}+s_i}{2}}^{\frac{s_i+s_{i+1}}{2}} (s_i - s^*) \, f(s^*) \, ds^* \, . \qquad (4.5.15)$$

From (3.2.39) we recognize that the term in front of the summation is the value of flexibility on d divided by the variance of s,

$$\mathcal{V}(Fd) = -\frac{t^2}{2q} \sigma^2 \, . \qquad (4.5.16)$$

Hence,

$$\mathcal{L}(D_d^M) = \mathcal{V}(Fd) \cdot L_2 \, , \qquad (4.5.17)$$

where

$$L_2 = \frac{1}{\sigma^2} \sum_{i=1}^{M} \int_{\frac{s_{i-1}+s_i}{2}}^{\frac{s_i+s_{i+1}}{2}} (s_i - s^*)^2 \, f(s) \, ds \, . \qquad (4.5.18)$$

OPTIMAL DISCRETIZATION

In view of (4.5.11), we will have an optimal discretization of the decision space if we choose the d_i so as to minimize

$$L_2 = \frac{1}{\sigma^2} \sum_{i=1}^{M} \int_{\frac{d_{i-1}+d_i}{2}}^{\frac{d_i+d_{i+1}}{2}} (d_i - d^*)^2 \, \{d^*|\mathcal{S}\} \, . \qquad (4.5.19)$$

Differentiating with respect to d_i and setting the result to zero, we obtain a set of necessary conditions for the optimal d_i:

$$d_i = \langle d^* \mid \frac{d_{i-1}+d_i}{2} < d^* < \frac{d_i+d_{i+1}}{2}, \mathcal{S} \rangle \, , \quad i = 1,\ldots,M \, . \qquad (4.5.20)$$

The optimal discretization points d_i are the conditional means of partitions whose boundaries are located halfway between the conditional means of the neighboring partitions. This is precisely the condition satisfied by the conditional means of the optimal quantizing intervals defined by (4.3.16). Therefore, the same procedures used in Section 4.3 for calculating good quantizing intervals may be used to calculate good discretizing schemes. In particular, the graphical technique is useful. The only difference between the two problems, of course, is that in the case of state variable quantizing it is the boundaries, the y_i in Fig. 4.6, that are important; in the case of decision discretization it is the conditional means, the x_i, that are important.

Table 4.4 gives the optimal discretizing points for the normalized uniform, exponential, and normal probability distributions. Table 4.5 gives values of L_2 for the three distributions under their respective optimal discretizations and for the equiprobable discretization in which $\mathrm{Prob}(d_{i-1} < d^* < d_i) = 1/M$ for all i, and $\mathrm{Prob}(d^* < d_1) = \mathrm{Prob}(d^* > d_M) = 1/2M$. The entries in the first column are identical to those of Table 4.3 since in the optimal case the expressions for L_1 and L_2 each reduce to

$$\min L_1 = \min L_2 = \frac{1}{2} \sum_{i=1}^{M} \int_{x_{i-1}}^{x_i} (x - \langle x \mid x_{i-1} < x < x_i, \mathcal{S} \rangle)^2 \{x \mid \mathcal{S}\} \, , \qquad (4.5.21)$$

where

$$x_i = \frac{\langle x \mid x_{i-1} < x < x_i, \mathcal{S} \rangle + \langle x \mid x_i < x < x_{i+1}, \mathcal{S} \rangle}{2} \, , \quad i = 1,\ldots,M \, . \qquad (4.5.22)$$

We reach the same qualitative conclusions about the behavior of L_2 as we did about L_1. The function has a flat minimum indicating that optimal discretizations need not be precisely specified. The function

TABLE 4.4

Optimal Discretizing Points for the Quadratic Problem

NUMBER OF LEVELS M	UNIFORM DISTRIBUTION $f(d^*) = \frac{1}{2}, -1 \le d^* \le 1$		EXPONENTIAL DISTRIBUTION $f(d^*) = e^{-d^*}, d^* \ge 0$		NORMAL DISTRIBUTION $f(d^*) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d^{*2}}{2}} \quad -\infty < d^* < \infty$	
	DECISION POINTS	DECISION PROBABILITIES	DECISION POINTS	DECISION PROBABILITIES	DECISION POINTS	DECISION PROBABILITIES
2	-.500 .500	.500 .500	.594 2.594	.797 .203	-.798 .798	.500 .500
3	-.667 .000 .667	.333 .333 .333	.424 1.611 3.611	.639 .288 .073	-1.224 .000 1.224	.270 .459 .270
4	-.750 -.250 .250 .750	.250 .250 .250 .250	.330 1.178 2.365 4.365	.530 .300 .135 .035	-1.510 -.453 .453 1.510	.164 .336 .336 .164
5	-.800 -.400 .000 .400 .800	.200 .200 .200 .200 .200	.270 .931 1.778 2.966 4.966	.451 .291 .165 .074 .019	-1.705 -.765 .000 .765 1.705	.109 .237 .307 .237 .109
6	-.833 -.500 -.167 .167 .500 .833	.167 .167 .167 .167 .167 .167	.229 .770 1.430 2.278 3.465 5.465	.393 .274 .176 .100 .045 .012	-1.896 -1.000 -.318 .318 1.000 1.896	.074 .181 .245 .245 .181 .074

TABLE 4.5

Values of the Discretizing Loss Factor L_2

DISTRIBUTION	DISCRETIZATION	M=2	3	4	5	6
Uniform	Optimal	.2500	.1111	.0625	.0400	.0278
	Equiprobable	.2500	.1111	.0625	.0400	.0278
Exponential	Optimal	.3542	.1797	.1090	.0731	.0522
	Equiprobable	.5560	.3748	.2828	.2271	.1897
Normal	Optimal	.3634	.1902	.1175	.0798	.0580
	Equiprobable	.3785	.2149	.1448	.1072	.0838

decreases rapidly with the level M of discretization up to about $M = 3$, telling us that most of the value of flexibility can be obtained through the use of very rough discretization.

4.6 CONCLUSIONS

Obviously a good deal more work has to be done before we gain a thorough understanding of the effects of quantization on state variables and discretization on decision variables. However, our initial results are quite encouraging as they seem to imply that the common technique of quantizing state variables is a highly effective method for evaluating and utilizing information. Similarly, discretizing decision variables seems to be a very practical method for evaluating flexibility.

CHAPTER 5

APPLICATION OF THE FLEXIBILITY CONCEPT TO DECISION MODEL DESIGN AND ANALYSIS

The flexibility concept and the quadratic results of Chapters 3 and 4 enable us to expand a useful technique for decision model design and analysis called "proximal analysis." The term proximal analysis is short for "approximate sensitivity analysis."

In Sections 5.1 and 5.2 a philosophy of model design is presented, and the usefulness of sensitivity analysis to this process is briefly discussed. The expansion of proximal analysis to include the concept of flexibility is the subject of Section 5.3.

5.1 DESIGN OF SIMPLIFIED MODELS

In previous chapters we repeatedly indicated that a decision maker may wish to use a less than complete decision model because of the high cost of complete decision analysis. Since there are a large number of alternative simplifications, or alternative models, the question arises as to how to choose the simplified model that is most appropriate for a given resource commitment problem.

In theory, the problem of choosing between competing models can be solved using the formal decision analysis technique. Such an approach involves defining a space of decision models, what Smallwood [12] refers to as the "metamodel," and then choosing from that space the particular model and corresponding decision strategy that yield the highest expected utility consistent with the beliefs and preferences of the decision maker. The problems associated with such a formulation, however, preclude its practical use.

Instead of using formal analysis, the process of model specification is usually approached in a heuristic manner. The basic technique is to propose a simplified model, using "rules of thumb" based upon experience, and then to improve that model until it appears to capture adequately the real system's perceived dynamics.

Therefore, a good portion of the model design process consists of what Demski [5, p. 32] refers to as determining the "scanning information" that will indicate improvements that should be incorporated into the model. By testing the current simplified model, the decision analyst can often generate much of the needed scanning information. One of the most useful test techniques is sensitivity analysis.

5.2 SENSITIVITY ANALYSIS

In sensitivity analysis the decision analyst tries to determine the change in the model's selection of alternative actions or expected utility that would result from a given change in the model's assumptions. Assumptions that produce small changes are apparently relatively insignificant, while assumptions that produce considerable changes are likely quite significant. Significant assumptions pertaining to model structure should, of course, be carefully investigated.

We shall illustrate the fundamental nature of sensitivity analysis with applications to the simplified analysis of information and flexibility.

INFORMATION EVALUATION USING SENSITIVITY ANALYSIS

Suppose a businessman is analyzing the possibility of expanding his product line. In determining the price he will charge for his new product, he may wonder how important it is for him to consider the wage settlement in a labor dispute that is currently being negotiated by one of his competitors. In other words, our businessman is wondering whether the wages that will be paid by his competitor will be an important consideration in planning his own pricing strategy.

In general, sensitivity analysis may be used to estimate the value of explicitly including the outcome of uncertain information variables into a model. Two basic sensitivity calculations are necessary for the analysis. The first is referred to as <u>open-loop sensitivity analysis</u>, and it addresses the question of how the value resulting from a fixed decision setting varies with changes in state variable settings. To illustrate, consider the basic decision model of Fig. 1.4. Suppose that the decision vector is set to its optimum, $\underline{d}^*$, and that the i^{th} state variable s_i is set to a fixed value $\hat{s}_i$. Under these conditions, the expected payoff of the resultant lottery is given by

$$\langle v|\hat{s}_i,\underline{d}^*,\mathcal{E}\rangle = \int_{\underline{s}\in S} \langle v|\underline{s},\underline{d}^*,\mathcal{E}\rangle \{\underline{s}|\hat{s}_i,\mathcal{E}\} . \tag{5.2.1}$$

Ranging $\hat{s}_i$ over some specific set of values indicates how the value function varies with the value of s_i . Howard [8] refers to this as open-loop sensitivity analysis because the decision is not altered as the information on the state variable is introduced.

The second calculation is referred to as <u>closed-loop sensitivity analysis</u>. It addresses the question of how the maximum value of the value function behaves as a function of state variable settings. If the i^{th} state variable has value $\hat{s}_i$, the maximum expected value of the lottery is

$$\langle v|\hat{s}_i,\underline{d}^*(\hat{s}_i),\mathcal{E}\rangle = \max_{\underline{d}\in D} \int_{\underline{s}\in S} \langle v|\underline{s},\underline{d},\mathcal{E}\rangle \{\underline{s}|\hat{s}_i,\mathcal{E}\} . \tag{5.2.2}$$

Varying $\hat{s}_i$ shows how the maximum payoff value varies with the setting of the i^{th} state variable.

Using the open-and closed-loop sensitivities the decision analyst may estimate the value to the decision maker of knowing the outcome of s_i . Taking the difference between the two quantities we obtain the value of <u>stochastic compensation</u>:

$$I(\hat{s}_i) = \langle v|\hat{s}_i,\underline{d}^*(\hat{s}_i),\mathcal{E}\rangle - \langle v|\hat{s}_i,\underline{d}^*,\mathcal{E}\rangle . \tag{5.2.3}$$

The value of stochastic compensation is the maximum amount that an expected value decision maker would be willing to pay in an uncertain environment to fix the i^{th} state variable at $\hat{s}_i$. Therefore, taking the expectation of (5.2.3) with respect to the distribution of s_i ,

$$\langle I|\mathcal{E}\rangle = \int_{s_i} I(s_i) \{s_i|\mathcal{E}\} , \tag{5.2.4}$$

we obtain the maximum amount the decision maker would pay for the revelation of s_i before setting his decision vector.

FLEXIBILITY EVALUATION USING SENSITIVITY ANALYSIS

The same ideas are involved in the analysis of whether to include a specific decision variable directly in the model. Suppose we wish to estimate the effect of obtaining flexibility on decision variable d_j . If we calculate the partially closed-loop sensitivity in which all decision variables but d_j are set to nominal or fixed optimal values $(\underline{d}-d_j)^*$ while d_j is allowed to vary optimally with $\underline{s}$, we get

$$\langle v|\underline{s},(\underline{d}-d_j)^*,d_j^*(\underline{s}),\mathcal{E}\rangle = \max_{d_j} \langle v|\underline{s},(\underline{d}-d_j)^*,d_j,\mathcal{E}\rangle . \tag{5.2.5}$$

Subtracting the open-loop sensitivity to the state vector $\underline{s}$, we obtain an expression for the value of partial stochastic compensation. This value is the maximum amount an expected value decision maker would be willing to pay to set the j^{th} decision variable to compensate for given state variable outcomes $\underline{s}$:

$$F(\underline{s}) = \langle v|\underline{s},(\underline{d}-d_j)^*,d_j^*(\underline{s}),\mathcal{E}\rangle - \langle v|\underline{s},\underline{d}^*,\mathcal{E}\rangle . \tag{5.2.6}$$

Taking the expectation over $\underline{s}$,

$$\langle F|\mathcal{E}\rangle = \int_{\underline{s}} F(\underline{s}) \; \{\underline{s}|\mathcal{E}\} , \tag{5.2.7}$$

we obtain the value to the decision maker of retaining flexibility on d_j , given complete information on the state variables.

APPROXIMATIONS IN SENSITIVITY ANALYSIS

The above analysis has illustrated that the value of information computation and the value of flexibility computation can be interpreted as calculations of expected compensation. The compensation function is defined as the difference between the model's open- and closed-loop sensitivities.

In the practical application of sensitivity analysis, approximate techniques are usually employed in the calculation of the open- and closed-loop sensitivities. Such approximations make it possible for the decision analyst to obtain, at relatively low cost, approximations to the value of information given flexibility. These, in turn, provide indications of the benefits of various information system and model design proposals and, therefore, help to guide the analyst in his model development. Some practical simplifications, including the ceteris paribus fashion in which sensitivity analysis is usually conducted, are illustrated in the following section.

5.3 PROXIMAL ANALYSIS

A major problem associated with a complex decision model is the evaluation of the value lottery $v(\underline{s},\underline{d})$, $\underline{s} \in S$, $\underline{d} \in D$. In many practical cases the number of variables is so large that the costs of direct evaluation and even computer simulation become prohibitive. Calculation of stochastic compensation functions in this case is not possible.

A method which is often useful for dealing with such problems, termed "proximal decision analysis," has been developed by Howard [8]. Rice [11] has shown that for the evaluation of complete information given complete flexibility the conditions of the proximal model allow deterministic compensation functions to be substituted for stochastic compensation functions. We shall show how this technique may be expanded to provide an estimation of the value of incomplete information and flexibility.

In its basic form proximal analysis assumes a model of the form of Fig. 1.4 with unconstrained continuous decision variables, a smooth value function, and a utility function that is approximately exponential with a small or moderate degree of risk aversion. Following Howard and Rice we imagine that the decision analysis cycle has yielded the following data. First, we suppose that the deterministic phase has resulted in a deterministic model relating outcome value to state and decision variable settings. A probabilistic phase is assumed to have supplied us with a vector of mean values $\overline{s}$ and a matrix of covariances among the state variables. Howard [8, p. 511] suggests a procedure for obtaining this data.

Let

$$\hat{\underline{d}} = \max_{\underline{d}}^{-1} v(\overline{\underline{s}},\underline{d}) \qquad (5.3.1)$$

denote the optimum deterministic decision setting and suppose that

$v(\underline{s},\underline{d})$ is approximated by a second-order Taylor series expansion about the point $(\bar{\underline{s}},\hat{\underline{d}})$. If state and decision variable settings are incremented by amounts $\Delta\underline{s}$ and $\Delta\underline{d}$ from $\bar{\underline{s}}$ and $\hat{\underline{d}}$, respectively, then the approximate increase in v, denoted Δv, is (see Section 3.2) given by

$$\Delta v = \underline{b}'\Delta\underline{s} + \frac{1}{2}\Delta\underline{s}'W\Delta\underline{s} + \Delta\underline{s}'T\Delta\underline{d} + \frac{1}{2}\Delta\underline{d}'Q\Delta\underline{d} \,, \tag{5.3.2}$$

where

$$\underline{b} = \left[\frac{\partial v}{\partial s_i}\bigg|_{(\bar{\underline{s}},\hat{\underline{d}})}\right] , \tag{5.3.3}$$

$$W = \left[\frac{\partial^2 v}{\partial s_i \partial s_j}\bigg|_{(\bar{\underline{s}},\hat{\underline{d}})}\right] , \tag{5.3.4}$$

$$T = \left[\frac{\partial^2 v}{\partial s_i \partial d_j}\bigg|_{(\bar{\underline{s}},\hat{\underline{d}})}\right] , \tag{5.3.5}$$

$$Q = \left[\frac{\partial^2 v}{\partial d_i \partial d_j}\bigg|_{(\bar{\underline{s}},\hat{\underline{d}})}\right] . \tag{5.3.6}$$

The linear term in $\underline{d}$ is not present since

$$\underline{r} = \left[\frac{\partial v}{\partial d_j}\bigg|_{(\bar{\underline{s}},\hat{\underline{d}})}\right] = 0 \,. \tag{5.3.7}$$

OPEN- AND PARTIALLY CLOSED-LOOP SENSITIVITIES

We wish now to find the open-loop sensitivity of v to changes in state variables s_i with i belonging to some index set I. The result is obtained from (5.3.2) with $\Delta\underline{d} = 0$, $\Delta s_k = 0$, $k \notin I$. Using the notation of Section 3.2,

$$\Delta v_{ol} = \underline{b}_I'\Delta\underline{s}_I + \frac{1}{2}\Delta\underline{s}_I'W_{II}\Delta\underline{s}_I \tag{5.3.8}$$

gives the open-loop sensitivity.

Next we shall calculate the partially closed-loop sensitivity in which the only decision variables that may be adjusted are those d_j with j in an index set J. Putting $\Delta\underline{s}_{\bar{I}}$ and $\Delta\underline{d}_{\bar{J}}$ equal to zero in (5.3.2),

$$\Delta v = \underline{b}_I' \Delta\underline{s}_I + \frac{1}{2} \Delta\underline{s}_I' W_{II} \Delta\underline{s}_I + \Delta\underline{s}_I' T_{IJ} \Delta\underline{d}_J + \frac{1}{2} \Delta\underline{d}_J' Q_{JJ} \Delta\underline{d}_J \,. \tag{5.3.9}$$

Setting the gradient with respect to $\underline{d}_J$ equal to zero, we get an expression showing how the flexible decision variables are optimally adjusted in response to changes in state variables:

$$\Delta\underline{d}_J^* = -Q_{JJ}^{-1} T_{IJ}' \Delta\underline{s}_I \,. \tag{5.3.10}$$

Substituting this expression into (5.3.9) gives the partially closed-loop sensitivity of outcome value to state variable changes:

$$\Delta v_{cl} = \underbrace{\underline{b}_I' \Delta\underline{s}_I + \frac{1}{2} \Delta\underline{s}_I' W_{II} \Delta\underline{s}_I}_{\text{open-loop sensitivity}} - \underbrace{\frac{1}{2} \Delta\underline{s}_I' T_{IJ} Q_{JJ}^{-1} T_{IJ}' \Delta\underline{s}_I}_{\text{effect of compensation}} \,. \tag{5.3.11}$$

We see, in analogy with Howard's results [8, Equation 7.4], that the partially closed-loop sensitivity is composed of terms representing the open-loop sensitivity to state variables plus terms that show the effect of compensation.

THE EXPECTED VALUE OF DETERMINISTIC COMPENSATION

Subtracting (5.3.8) from (5.3.11) we get an expression for the value of deterministic compensation for the quadratic decision problem:

$$v_{comp}(\Delta\underline{s}_I) = -\frac{1}{2} \Delta\underline{s}_I' T_{IJ} Q_{JJ}^{-1} T_{IJ}' \Delta\underline{s}_I \,. \tag{5.3.12}$$

The value of deterministic compensation tells us what it would be worth to the decision maker in a completely deterministic environment to adjust decision variables in response to changes in state variables.

The expected value of deterministic compensation is obtained by taking the expectation of (5.3.12) with respect to the marginal probability distribution of s_I:

$$\langle v_{comp} | \mathcal{E} \rangle = -\frac{1}{2} E(\Delta\underline{s}_I' T_{IJ} Q_{JJ}^{-1} T_{IJ}' \Delta\underline{s}_I) = -\frac{1}{2}\text{trace}[T_{IJ} Q_{JJ}^{-1} T_{IJ}' E(\Delta\underline{s}_I \Delta\underline{s}_I')] . \tag{5.3.13}$$

A comparison with (3.2.39) shows that (5.3.13) is exactly the expected value of perfect information on $\underline{s}_I$ given perfect flexibility on $\underline{d}_J$ for an expected value decision maker with a quadratic value function if any of the conditions of Corollary 2 are satisfied.

Now suppose that all state variables are adjusted in the sensitivity calculations. The compensation function becomes

$$v_{comp}(\Delta\underline{s}) = \Delta\underline{s}' T_{NJ} Q_{JJ}^{-1} T_{NJ}' \Delta\underline{s} . \tag{5.3.14}$$

If the function $E(\Delta\underline{s} | \Delta\underline{s}_I)$ is available, the compound function

$$v_{comp}[E(\Delta s | \Delta s_I)] = E(\Delta\underline{s}' | \Delta\underline{s}_I) T_{NJ} Q_{JJ}^{-1} T_{NJ}' E(\Delta\underline{s} | \underline{s}_I) \tag{5.3.15}$$

may be formed. Taking the expectation of (5.3.15) yields

$$\langle v_{comp} | \mathcal{E} \rangle = E[E(\Delta\underline{s} | \Delta\underline{s}_I) T_{NJ} Q_{JJ}^{-1} T_{NJ}' E(\Delta\underline{s} | \Delta\underline{s}_I)] , \tag{5.3.16}$$

which is the expected value of perfect information on $\underline{s}_I$ given perfect flexibility on $\underline{d}_J$ for an expected value decision maker with a quadratic value function if any of the conditions of Corollary 1 are satisfied.

APPROXIMATING THE EVPIGPF WITH SENSITIVITY ANALYSIS

Howard [8, Appendix B] gives a method for numerically evaluating $\underline{b}$, W , T , Q , and various conditional and unconditional covariance matrices. Hence, the proximal model and the theorem and corollaries of Chapter 3 provide a means for obtaining an approximation to the expected value of information given flexibility for the risk sensitive decision maker with a smooth value function.

The above results, however, show that under certain conditions a simpler procedure may be applied. For the purpose of illustration, assume that the value function for the decision model contains two state variables and two decision variables. We wish to estimate the value of perfect information on s_1 given perfect flexibility on d_2. For the first calculation we shall ignore the effect that knowledge of s_1 has on the estimation of s_2. The procedure consists of

1. evaluating deterministic open-loop sensitivity to changes in the observable state variable s_1,
2. evaluating deterministic partially closed-loop sensitivity (d_2 continuously optimized) to changes in s_1,
3. calculating the difference in these two functions, $v_{comp}(\Delta s_1)$,
4. determining the expectation of v_{comp}.

If knowledge of s_1 impacts the decision through its effect on the estimation of s_2, this may be included in the approximation using the following procedure:

1. evaluate deterministic open-loop joint sensitivity to changes in s_1 and s_2,
2. evaluate deterministic partially closed-loop joint sensitivity (d_2 continuously optimized) to changes in s_1 and s_2,
3. calculate the difference in these two functions, $v_{comp}(\Delta s_1, \Delta s_2)$,
4. determine $E(\Delta s_2 | \Delta s_1)$, the conditional mean of Δs_2 as a function of Δs_1,
5. determine the expected value of $v_{comp}[\Delta s_1, E(\Delta s_2 | \Delta s_1)]$.

Implementation of this procedure could be facilitated by approximating joint sensitivities by quadratic functions. A good approximation may be expected provided that $E[E(\Delta s_2 | \Delta s_1)] = 0$; that is, the prior expectation is a zero shift in the mean of the unobservable state variable.

CHAPTER 6

SUMMARY, CONCLUSIONS, AND SUGGESTIONS FOR FURTHER RESEARCH

6.1 SUMMARY AND CONCLUSIONS

The objective of the thesis has been to demonstrate that the concept of decision flexibility may be usefully incorporated into the decision analysis framework. The demonstration has been based on one possible definition of flexibility — the mathematically precise definition that was presented in Chapter 1.

We feel confident that we have achieved at least some measure of success. In Chapter 1, application to the classical problem of choosing plant flexibility indicated that our definition provides an analytic representation of flexibility that seems to preserve successfully intuitive notions. A logical and consistent means for calculating the value of any combination of information and flexibility was presented in Chapter 2. By applying the technique to a sample decision problem, we demonstrated that the concept might be useful for generating insights into problem structure. It was shown that just as the standard value of information calculation could be used to evaluate information-gathering schemes, the value of information given flexibility could be used to evaluate information distribution and utilization systems.

An exploration into the value of information given flexibility for quadratic decision problems was undertaken in Chapter 3. We obtained a closed-form expression for the value of perfect information on a subset of state variables given perfect flexibility on a subset of decision variables. Alternatively, this result may be regarded as the solution to the two-stage quadratic decision problem in which some decisions must be made now, while others may be delayed until after the receipt of some information. The relation of the value of joint information to the sum of individual values of information was clarified by the quadratic analysis. It was shown that the first order additivity or non-additivity of the value of information is determined by state variable correlation. To a first order approximation, if two pieces of information are uncorrelated, then the value of obtaining that information simultaneously equals the sum of the values of receiving each item of information by itself. Similarly, the first order determinant of the additivity or non-additivity of the value of flexibility is decision variable interaction. If the value function is additive in two decision vectors, then to first order, the value of simultaneously obtaining flexibility on both decision vectors will equal the sum of the values of obtaining flexibility on each vector individually.

The results concerning quantization in Chapter 4 were indicative of an important "rule of thumb" for decision analysis. Most of the value of including a state information variable or a flexible decision into a decision model is achieved by using three or more levels of quantization in a decision tree. The marginal gain of using more than four levels seems to be small. The effect, while relatively clear for independent state variables and non-interacting decision variables, tends to be obscured if the state variables are highly correlated or if the decision variables interact strongly with other variables in the model.

Our results demonstrated that once the number of quantizing and discretizing levels has been chosen, the sensitivity to the particular quantizing scheme is relatively small. If practical considerations indicate that considerable importance is associated with small differences in certain ranges of a state or decision variable, then, certainly, such information should be taken into account. In the absence of such information the graphical technique of Section 4.3 may be employed. A good rule of thumb for quantizing a probability distribution is to use roughly equal probability portioning with regions of high probability density getting slightly more probability than regions of low probability density.

The computational difficulty of performing the value of information given flexibility calculations gives impetus to a search for simplifying approximations. The analysis of Chapter 5 showed that an approximation to the expected value of perfect information given perfect flexibility could be obtained by applying sensitivity analysis to the decision problem's deterministic value model. This enables a deterministic compensation function to be evaluated. This function is then combined with the conditional expectation of the state vector $\underline{s}$

given the observable state variables $\underline{s}_I$. The approximation to the expected value of information given flexibility is obtained by integrating this combined function over the prior distribution of the observable state variables.

6.2 SUGGESTIONS FOR FURTHER RESEARCH

In the course of our work a number of what appear to be highly productive avenues for research were identified. Unfortunately, because of time constraints these avenues were only briefly explored. We mention here two main areas where we feel additional investigation would prove highly fruitful.

Our definition of flexibility has been shown to provide insight and understanding into several problems. Application of our definition (or similar definitions of decision flexibility) to other common decision problems would likely provide similar problem insights. Consider, for example, an analysis of flexibility in the problem of liquidity preference. Results might indicate problem characteristics that determine the logical quantity of resources to be held in reserve for the purpose of meeting unexpected opportunities or requirements. Such results would prove highly useful. For this reason, we feel that exploration into the concept of decision flexibility should be continued.

Secondly, there seems to be potential for the analysis of the economic costs of discretization and quantization. Analytic techniques for investigating the distortion caused by data quantization are nearly as plentiful as the use of quantized data systems. We have shown that such data distortion can be evaluated in precise economic terms. Such a result provides the logical means for judging a quantizing scheme and an objective function for the design of optimal information reporting and data processing systems.

APPENDIX A

SUMMARY OF BASIC NOTATION

$\underline{s}$ = $(s_1,\ldots,s_n)$ = a random vector describing the state of Nature.

S = the set of conceivable states $\underline{s}$ of Nature. Uncertainty about $\underline{s}$ is expressed by a probability distribution defined on S.

$\mathcal{S}$ = a state of experience or information on which probability assignments are made.

$\mathcal{E}$ = the experience level prior to the analysis.

$\{\underline{s}|\mathcal{S}\}$ = the probability density function describing the uncertainty in $\underline{s}$ when the state of information is $\mathcal{S}$.

$\langle\underline{s}|\mathcal{S}\rangle = \int_{\underline{s}\in S} \underline{s}\{\underline{s},\mathcal{S}\}$ = the expected value of $\underline{s}$ given $\mathcal{S}$.

$\underline{d}$ = $(d_1,\ldots,d_m)$ = a vector of decision or action variables over which a decision maker has direct control.

$\underline{d}-d_j$ = the vector composed of all decision variables but d_j.

D = the set of conceivable actions.

Y = a set of possible information signals y.

η = an information structure (a function from S to Y) which produces the information signal $y = \eta(\underline{s})$ when $\underline{s}$ is the true state of Nature.

ζ = $(\zeta_1,\ldots,\zeta_m)$ = an information distribution structure (a vector of functions defined on Y). $\zeta_j(y)$ is the information made available for the setting of the j^{th} decision variable when the information signal is y.

$\underline{d}(\cdot) = (d_1(\cdot),\ldots,d_m(\cdot))$ = a decision strategy (a function of y, $\zeta(y)$, or the level of experience $\mathcal{S}$, whose domain is D). If danger of confusion is minimal the function may be abbreviated $\underline{d}$.

$\mathcal{D}$ = the set of feasible decision strategies.

v = a value function (a real valued function defined on $S \times R$). The monetary value to the decision maker if he chooses action $\underline{d}$ and the state of Nature is $\underline{s}$, is $v(\underline{s},\underline{d})$.

u = a utility function (a real valued function defined on R). The utility to the decision maker of the monetary value v is $u(v)$.

π = a profit function.

N = $\{1,\ldots,n\}$ = the set of state variable indices.

M = $\{1,\ldots,m\}$ = the set of decision variable indices.

I = a subset of N denoting the set of indices corresponding to those state variables on which information is available. $\bar{I}$ denotes the complement of I.

J = a subset of M denoting the set of indices of decision variables for which flexibility is available. $\bar{J}$ denotes the complement of J.

T_{IJ} = the matrix $[t_{kl}]_{k\epsilon I, l\epsilon J}$ of those elements t_{kl} of the matrix T such that k is in I and l is in J.

$\underline{s}_I$ = the vector of state variables on which perfect information is available.

$\underline{x}$ = $E(\underline{s}|\underline{s}_I)$ = the conditional mean of $\underline{s}$ given perfect information on $\underline{s}_I$.

$\underline{d}_J$ = the vector of decision variables for which perfect flexibility is available.

$C\underline{s}_I F\underline{d}_J$ = the information structure for which perfect information is available on $\underline{s}_I$ and perfect flexibility exists on $\underline{d}_J$.

$V_{C\underline{s}_I F\underline{d}_J}$ = the value of the information structure $C\underline{s}_I F\underline{d}_J$

APPENDIX B

THE ENTREPRENEUR'S PRICE-QUANTITY DECISION

We present here the solution to the entrepreneur's problem of Section 2.2. An entrepreneur plans to produce a new product and wishes to decide upon a price and quantity that will maximize his expected profits. His problem would not be too difficult if it were not for uncertainty, but he recognizes that he is uncertain about what his costs will be and about what the demand for his product will be.

We suppose that our entrepreneur feels that the demand for his product will be related to the price he charges according to

$$x = \frac{a}{p} - b - e , \tag{B.1}$$

where

x = demand (in thousands of units) ,

p = price per unit (in thousands of dollars) ,

e = a random variable uniformly distributed from zero to one ,

a,b = constants .

Suppose, further, that he feels his costs c per unit are well represented by a random variable uniformly distributed between zero and one:

c = total cost per unit (in thousands of dollars) .

We assume the random variables e and c are independently distributed. The demand curve (B.1) and the probability density function for the demand parameter e are illustrated in Fig. 2.4. The probability density function for the cost variable c is illustrated in Fig. 2.3.

The objective of our analysis is to determine our entrepreneur's optimal price and quantity and the corresponding expected net profit under various perfect information given perfect flexibility structures. Let

q = quantity of product produced (in thousands of units) ,

v = net profit (in millions of dollars) .

Then,

$$v(p,q,c,e) = \begin{cases} p(\frac{a}{p} - b - e) - cq \ , & \text{if } \frac{a}{p}\text{-}b\text{-}e < q \\ (p - c)\, q \ , & \text{if } \frac{a}{p}\text{-}b\text{-}e > q \end{cases} \quad . \tag{B.2}$$

Suppose first that our entrepreneur seeks no further information. As a function of price and quantity his expected net profit will be

$$\langle v|p,q,\mathcal{E}\rangle = \int_c \int_e v(p,q,c,e) \ \{e|\mathcal{E}\} \ \{c|\mathcal{E}\}$$

$$= a - \frac{a^2}{2p} - (b + \frac{b^2}{2} + \frac{1}{2})p + ba + (a - \frac{1}{2})q - pbq - \frac{pq^2}{2} \ . \tag{B.3}$$

Differentiating and setting the result equal to zero we obtain expressions for the optimal price and quantity,

$$p^* = \sqrt{\frac{4a-1}{8b+4}} \quad , \quad q^* = \sqrt{\frac{(2a-1)^2(2b+1)}{4a-1}} - b \ , \tag{B.4}$$

which yield a maximal expected profit of

$$\langle v|\mathcal{E}\rangle = \langle v|p^*,q^*,\mathcal{E}\rangle = a + \frac{b}{2} - \frac{1}{2}\sqrt{(2b+1)(4a-1)} \quad . \tag{B.5}$$

To be specific, let us suppose that $a = 2.25$, $b = .5$. Then

$$p^* = 1 = \$1{,}000 \ , \quad q^* = 1.25 = 1{,}250 \text{ units} \ . \tag{B.6}$$

The expected demand is

$$\langle x|\mathcal{E}\rangle = \frac{a}{p^*} - b - \langle e|\mathcal{E}\rangle = 1.25 = 1{,}250 \text{ units} \ , \tag{B.7}$$

and the expected net profit is

$$\langle v|\mathcal{E}\rangle = .5 = \$500{,}000 \ . \tag{B.8}$$

We shall now proceed to calculate the optimal decision strategies, the expected payoffs, and the values of the several information-distribution structures in which perfect information is available on various state variables and flexibility exists on various decision variables. In all there are nine possible distinct combinations of state variable information and decision variable flexibility. Unless otherwise noted, in each of these cases the optimal decision strategy is found by differentiation. By (2.1.7), the values of the information-flexibility structures (the entries of Table 2.1) will be the differences between the expected profit under the structures and Eq. (B.7).

<u>Perfect Information on c and e -- Perfect Flexibility on p and q</u>

If costs and demand are revealed to the entrepreneur before he sets his price and quantity, then, obviously, he can expect to increase his profits. In such a situation the maximal expected profit

$$\langle v|CceFpq,\mathcal{E}\rangle = \int_c \int_e \max v(p,q,c,e)\ \{e|\mathcal{E}\}\ \{c|\mathcal{E}\}$$

$$= a + \frac{b}{2} + \frac{1}{4} - \frac{8}{9}\, a^{1/2}[(b+1)^{3/2} - b^{3/2}] \tag{B.9}$$

is obtained by setting

$$p^* = \sqrt{\frac{ca}{b+e}}\ , \qquad q^* = \sqrt{\frac{(b+e)a}{c}} - b - e \tag{B.10}$$

(assuming that $a > b + e$ so that $p^* > c$). With $a = 2.25$, $b = .5$, $\langle v|CceFpq,\mathcal{E}\rangle = \$771{,}915$.

<u>Perfect Information on c -- Perfect Flexibility on p and q</u>

Should costs alone be revealed prior to the setting of p and q,

$$\langle v|CcFpq,\mathcal{E}\rangle = \int_c \max_{p,q} \int_e v(p,q,c,e)\ \{e|\mathcal{E}\}\ \{c|\mathcal{E}\}$$

$$= a + \frac{b}{2} - \sqrt{2b+1} \int_0^1 \sqrt{2ac-c^2}\ dc$$

$$= a + \frac{b}{2} - \frac{\sqrt{2b+1}}{2}\left[(1-a)\sqrt{2a-1} + a^2\left(\sin^{-1}\frac{1-a}{a} + \frac{\pi}{2}\right)\right] \tag{B.11}$$

is attained with

$$p^* = \sqrt{\frac{(2a-c)c}{2b+1}} \quad , \quad q^* = (a-c)\sqrt{\frac{2b+1}{(2a-c)c}} - b \ . \qquad \text{(B.12)}$$

With the example values $a = 2.25$, $b = .5$, $\langle v|CcFpq,\mathcal{E}\rangle = \$639,416$.

Perfect Information on e -- Perfect Flexibility on p and q

Should the demand parameter e be revealed prior to the setting of p and q ,

$$\langle v|CeFpq,\mathcal{E}\rangle = \int_{e} \max_{p,q} \int_{c} v(p,q,c,e)\ \{c|\mathcal{E}\}\ \{e|\mathcal{E}\}$$

$$= a + \frac{b}{2} + \frac{1}{4} - \frac{(8a)^{1/2}}{3}\left[(b+1)^{3/2} - b^{3/2}\right] , \qquad \text{(B.13)}$$

and the optimal decision strategy is

$$p^* = \sqrt{\frac{a}{2(b+e)}} \quad , \quad q^* = \sqrt{2a(b+e)} - b - e \ . \qquad \text{(B.14)}$$

With $a = 2.25$, $b = .5$, $\langle v|CeFpq,\mathcal{E}\rangle = \$651,924$.

Perfect Information on c and e -- Perfect Flexibility on p

If perfect information on costs and demand is used only for the setting of price,

$$\langle v|CceFp,\mathcal{E}\rangle = \max_{q} \int_{c} \int_{e} \max_{p} v(p,q,c,e)\ \{e|\mathcal{E}\}\ \{c|\mathcal{E}\}$$

$$= \max_{q} \left(-\frac{q}{2} + aq \log \frac{q+b+1}{q+b}\right) . \qquad \text{(B.15)}$$

The optimal pricing strategy sets p at the highest value that will sell out the entire production quantity.

$$p^* = \frac{a}{q+b+e} \ . \qquad \text{(B.16)}$$

With the parameter values $a = 2.25$, $b = .5$, the maximizing q in

(B.15) is approximately $q^* = 1.0947$, and this yields an expected profit of $\langle v|CceFp,\mathcal{E}\rangle = \$651,639$.

Unlike the other optimal strategies, if our entrepreneur chooses this information-flexibility structure, it is conceivable that he will lose money! In the worst possible outcome, demand will be at its lowest ($e = 1$) and costs will be at their highest ($c = 1$) conceivable values. The pricing strategy (B.16) puts $p^* = \$867.15$: our entrepreneur will lose \$132.85 on each of the 1,095 units he sells.

Perfect Information on c and e -- Perfect Flexibility on q

If perfect information on costs and demand is used only for the setting of quantity,

$$\langle v|CceFq,\mathcal{E}\rangle = \max_p \int_c \int_e \max_q v(p,q,c,e)\ \{e|\mathcal{E}\}\ \{c|\mathcal{E}\}$$

$$= a + \frac{b}{2} + \frac{1}{4} - \sqrt{a(2b+1)}\ , \qquad \text{(B.17)}$$

$$p^* = \sqrt{\frac{a}{2b+1}}\ , \quad q^* = \sqrt{a(2b+1)} - b - e\ . \qquad \text{(B.18)}$$

It is assumed in (B.17) and (B.18) that $a > 2b+1$ so that costs may not exceed p^* . With the values $a = 2.25$, $b = .5$, $\langle v|CceFq,\mathcal{E}\rangle =$ \$628,680 .

Perfect Information on c -- Perfect Flexibility on p

Suppose our entrepreneur will have perfect information on costs but uses that information only in the setting of his price. Then,

$$\langle v|CcFp,\mathcal{E}\rangle = \max_q \int_c \max_p \int_e v(p,q,c,e)\ \{e|\mathcal{E}\}\ \{c|\mathcal{E}\}$$

$$= a + \frac{b}{2} - \frac{1}{2}\sqrt{(2b+1)(4a-1)}\ , \qquad \text{(B.19)}$$

$$p^* = \sqrt{\frac{4a-1}{8b+4}}\ , \quad q^* = (2a-1)\sqrt{\frac{2b+1}{4a-1}}\ . \qquad \text{(B.20)}$$

Comparison of (B.19) and (B.20) with (B.5) and (B.4) shows that knowledge of c does not aid in the setting of p. While knowing costs will improve our entrepreneur's estimate of his expected profits, it does not help him to improve his pricing strategy. For this reason there is no economic advantage to seeking information on costs if this information cannot be used for the setting of quantity.

<u>Perfect Information on c -- Perfect Flexibility on q</u>

Suppose the information on costs is used only for the setting of quantity. Then,

$$\langle v|CcFq,\mathcal{E}\rangle = \max_p \int_c \max_q \int_e v(p,q,c,e)\ \{e|\mathcal{E}\}\ \{c|\mathcal{E}\}$$

$$= \frac{3}{8}\,\frac{a^2}{(3b+2)}\ , \qquad \text{(B.21)}$$

$$p^* = \frac{3a}{6b+4}\ , \quad q^* = \begin{cases} 0\ , & \text{if } c > \dfrac{3a}{6b+4} \\ \dfrac{6b+4}{3a}(a-c) - b - c\ , & \text{otherwise} \end{cases}\ . \qquad \text{(B.22)}$$

Since quantity may be conditioned upon costs, if costs turn out to exceed his selling price the entrepreneur chooses not to produce his product. With the example values of $a = 2.25$, $b = .5$, (B.21) becomes $\langle v|CcFq,\mathcal{E}\rangle = \$542,411$.

<u>Perfect Information on e -- Perfect Flexibility on p</u>

Should our entrepreneur anticipate knowing demand before he sets his price, his expected profit is

$$\langle v|CeFp,\mathcal{E}\rangle = \max_q \int_e \max_p \int_c v(p,q,c,e)\ \{c|\mathcal{E}\}\ \{e|\mathcal{E}\}$$

$$= \max_q \left(-\frac{q}{2} + aq \log \frac{q+b+1}{q+b}\right)\ , \qquad \text{(B.23)}$$

and his optimal pricing strategy is

$$p^* = \frac{a}{q+b+e} \quad . \qquad (B.24)$$

These are the same results we obtained in the calculations under the structure CceFp . We conclude, therefore, that of the knowledge c , e , it is only the knowledge of e that has economic value.

Perfect Information on e -- Perfect Flexibility on q

Lastly, we suppose that our entrepreneur will receive perfect information on demand but may use that information only for his setting of quantity. Then,

$$\langle v|CeFq,\mathcal{E}\rangle = \max_p \int_e \max_q \int_c v(p,q,c,e)\ \{c|\mathcal{E}\}\ \{e|\mathcal{E}\}$$

$$= a + \frac{b}{2} + \frac{1}{4} - \sqrt{a(2b+1)} \ , \qquad (B.25)$$

$$p^* = \sqrt{\frac{a}{2b+1}} \ , \quad q^* = \sqrt{a(2b+1)} - b - e \ . \qquad (B.26)$$

This corresponds to (B.17) and (B.18). As far as setting quantity is concerned, the economic value of knowledge of costs and demand is achieved through knowledge of demand alone.

LIST OF REFERENCES

[1] Baumol, W. J., Economic Dynamics, 2nd ed., The Macmilian Company, New York, 1959, pp. 92-93.

[2] Bofinger, E., "Maximizing the Correlation of Grouped Observations," Journal of the American Statistical Association, Vol. 65, December 1970, pp. 1632-38.

[3] Cox, D. R., "Note on Grouping," Journal of the American Statistical Association, Vol. 52, December 1957, pp. 543-47.

[4] Dalenius, T., Sampling in Sweden, Stockholm: Almqvist and Wiksell, 1957, Chapter 7.

[5] Demski, J. S., Information Analysis, Addison-Wesley, Menlo Park, Calif., 1972.

[6] Hart, A. G., "Risk, Uncertainty and the Unprofitability of Compounding Probabilities," Studies in Mathematical Economics and Econometrics, O. Lange, F. McIntyre, and F. Intema, eds., The University of Chicago Press, Chicago, 1942.

[7] Howard, R. A., "The Foundations of Decision Analysis," IEEE Transactions on Systems Science and Cybernetics, Vol. SCC-4, No. 3, September 1968, pp. 211-219.

[8] Howard, R. A., "Proximal Decision Analysis," Management Science, Vol. 17, No. 9, May 1971.

[9] IEEE Transactions of System Science and Cybernetics, Special Issue on Decision Analysis, Vol. SCC-4, No. 3, September 1968.

[10] Marschak, T. and R. Nelson, "Flexibility, Uncertainty, and Economic Theory," Metroeconomica, April-December 1962, pp. 42-58.

[11] Rice, T. R., "Economics of Decision Making," Ph.D. Dissertation, Stanford University, 1974.

[12] Smallwood, R. D., "A Decision Analysis of Model Selection," IEEE Transactions on Systems Science and Cybernetics, Vol. SCC-4, No. 3, September 1968, pp. 333-42.

[13] Stigler, G., "Production and Distribution in the Short Run," Journal of Political Economy, June 1939, pp. 305-28.

[14] Theil, H., Economic Forecasts and Policy, 2nd ed., North Holland Publishing Company, Amsterdam, 1961, p. 372.

[15] Tisdell, C. A., The Theory of Price Uncertainty, Production, and Profit, The Princeton University Press, Princeton, 1968, Chapter 6.